Algorithms to Compute Topological Invariants of Subschemes of Smooth Toric Varieties

Martin Helmer

Department of Mathematics,
University of California, Berkeley

martin.helmer@berkeley.edu

Webpage: https://math.berkeley.edu/~mhelmer/
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Overview

We discuss algorithms to compute some topological invariants (in particular characteristics classes of schemes) for subschemes of certain smooth complete toric varieties X_Σ. We work over k, an algebraically closed field of characteristic zero.

- By a variety we shall mean the zeros of some set of polynomial equations.
- For the purposes of this talk one may think of a scheme as a more general algebraic variety and one may, if one prefers, freely substitute the word variety in place of scheme.
- We begin with probabilistic algorithms to compute the Euler characteristic and the Chern-Schwartz-MacPherson and Segre class of V a subscheme of $\mathbb{P}^n$.

Overview

- The more general setting where V is a subscheme of X_Σ will be discussed later in the talk.
- All these methods can be (and are) implemented symbolically using Gröbner bases calculations, or numerically using homotopy continuation.
- The algorithms are tested on several examples and are found to perform favourably compared to other existing algorithms.

Main Questions

- How can abstractly (ex. functorially) defined topological invariants (in particular characteristics classes) of algebraic varieties be realized in a concrete way?
- How can a concrete understanding of such invariants be leveraged to provide simple and effective methods for their computation?
- In what settings are such approaches to characteristics classes possible?

The topological Euler characteristic

Applications and Computation

- The Euler characteristic χ is an important topological invariant.
- Some recent applications include:
  - Maximum likelihood estimation in algebraic statistics by Huh [10] and by Rodriguez and Wang [14].
  - Applications to string theory in physics by Collinucci et al. [7] and by Aluffi and Esole [5].
- For projective schemes it can be computed several different ways, for example from Hodge numbers, or as we do here, from the Chern-Schwartz-Macperhson class.
- In particular from $c_{SM}(V)$ we may immediately obtain the Euler characteristic of V, since $\chi(V)$ is equal to the degree of the zero dimensional component of $c_{SM}(V)$.

The Setting for Characteristic classes in $\mathbb{P}^n$

- The Chow ring of $\mathbb{P}^n$ is $\oplus_{j=0}^{n} A^j(\mathbb{P}^n)$ where $A^j(\mathbb{P}^n)$ is the group of codimension j-cycles modulo rational equivalence.
- We have that $A^*(\mathbb{P}^n) \cong \mathbb{Z}[h]/(h^{n+1})$ where h is the rational equivalence class of a hyperplane in $\mathbb{P}^n$.
- A hypersurface W of degree d will be represented by $[W] = d \cdot h$ in $A^*(\mathbb{P}^n)$.
- We consider the c_{SM} class, the Segre class and other characteristic classes as elements of $A^*(\mathbb{P}^n)$ (more specifically we consider their respective pushforwards in $A^*(\mathbb{P}^n)$).

Chern-Schwartz-MacPherson Classes

- The c_{SM} class generalizes the Chern class of the tangent bundle to singular varieties/schemes, i.e. $c(T_V) \cdot [V] = c_{SM}(V)$ when V is a smooth subscheme of a smooth variety M.
- The c_{SM} class has important functorial properties and relations to the Euler characteristic. Specifically the c_{SM} class is the unique natural transformation between the constructible function functor and the Chow group functor.
- The Chern-Schwartz-MacPherson class has also been directly related to problems in string theory in Aluffi and Esole [4].
- For subschemes of $\mathbb{P}^n$ the relationship between the c_{SM} class and the Euler characteristic gives us a more concrete way to understand the information contained in the c_{SM} class.

Chern-Schwartz-MacPherson Classes

- When V is a subscheme of $\mathbb{P}^n$ the class $c_{SM}(V)$ can be thought of as a more refined version of the Euler characteristic since it contains the Euler characteristics of V and those of general linear sections of V for each codimension.
- Specifically (see Aluffi [3]), if $\dim(V) = m$, starting from $c_{SM}(V)$ we may directly obtain the list of invariants

$$\chi(V), \chi(V \cap L_1), \chi(V \cap L_1 \cap L_2), \ldots, \chi(V \cap L_1 \cap \cdots \cap L_m)$$

  where $L_1, \ldots, L_m$ are general hyperplanes.
- Alternatively one may obtain $c_{SM}(V)$ from the list of Euler characteristics above as well, i.e. there is an involution.

Example: c_{SM} Class and Euler Characteristics

- Consider the variety $V = V(x_0x_3 - x_1x_2)$ in $\mathbb{P}^3 = \mathrm{Proj}(k[x_0, \dots, x_3])$ which is the image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$.
- We may compute $c_{SM}(V) = 4h^3 + 4h^2 + 2h \in A^*(\mathbb{P}^3) \cong \mathbb{Z}[h]/(h^4)$ and obtain the Euler characteristics of the general linear sections using an involution formula given by Aluffi in [3], specifically:
    - First consider the polynomial $p(t) = 4 + 4t + 2t^2 \in \mathbb{Z}[t]/(t^4)$ given by the coefficients of the c_{SM} class above.
    - Next apply Aluffi's involution

$$p(t) \mapsto \mathcal{I}(p) := \frac{t \cdot p(-t-1) + p(0)}{t+1} = 2t^2 - 2t + 4.$$

- This gives $\chi(V) = 4$, $\chi(V \cap L_1) = 2$, and $\chi(V \cap L_1 \cap L_2) = 2$.

Segre Class of a Subscheme

- For V a proper closed subscheme of a variety W the Segre class $s(V, W)$ is the class of a particular vector bundle (specifically the normal cone $C_V W$).
- Can also write $s(V, W) = \eta_* \left(\frac{[\tilde{V}]}{1 + [\tilde{V}]} \right) \in A^*(V)$, where $\tilde{V}$ is the exceptional divisor of $Bl_V W$, the blowup of W along V and $\eta : \tilde{V} \to V$ is the projection.
- Can be used to define several other characteristic classes in some settings (e.g. Chern, Chern-Fulton, c_{SM}, and others).
- For subschemes of the smooth complete toric varieties we consider here we will give a more explicit (and easier to compute) expression in terms of the projective degrees below (we will see the $\mathbb{P}^n$ case soon).

Previous Algorithms to Compute Segre Classes

- The first algorithm to compute the Segre class $s(V, \mathbb{P}^n)$ for $V = V(I)$ a subscheme of $\mathbb{P}^n$ was that of Aluffi [2].
  - The algorithm of Aluffi works by computing the blowup $Bl_V\mathbb{P}^n$ of $\mathbb{P}^n$ along V (or equivalently by computing the Rees algebra of I).
  - This is a quite expensive computation in most cases.
- Another algorithm to compute $s(V, \mathbb{P}^n)$ is that of Eklund, Jost and Peterson [8].
  - This method is probabilistic and computes the Segre class by computing the degrees of certain residual sets via a particular saturation.
  - Specifically, the residual sets are those of Fulton's Residual Intersection Theorem.
- We will instead use the *projective degrees*.

Projective Degrees

- Consider a rational map $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$.
- We may define the *projective degrees* of the map ϕ as a list of integers $(g_0, \ldots, g_n)$ where

$$g_i = \mathrm{card}\left(\phi^{-1}\left(\mathbb{P}^{m-i}\right) \cap \mathbb{P}^i\right).$$

  - Here $\mathbb{P}^{m-i} \subset \mathbb{P}^m$ and $\mathbb{P}^i \subset \mathbb{P}^n$ are general hyperplanes of dimension $m - i$ and i respectively.
  - Additionally card is the cardinality of a zero dimensional set.
  - Consider the projective degrees arising from the rational map defined by an ideal.
  - $I = (f_0, \ldots, f_m)$ be a homogeneous ideal in $R = k[x_0, \ldots, x_n]$ and consider $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$, where $\phi : p \mapsto (f_0(p) : \cdots : f_m(p))$,

A New Expression for the Projective Degrees (1)

Theorem 1

- *Let k be an algebraically closed field and let $I = (f_0, \ldots, f_m)$ be a homogeneous ideal in $R = k[x_0, \ldots, x_n]$ defining a ϱ-dimensional scheme $V = V(I)$.*
- *Without loss of generality, assume that all the polynomials f_i generating I have the same degree d.*

The projective degrees $(g_0, \ldots, g_n)$ of $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$, where $\phi : p \mapsto (f_0(p) : \cdots : f_m(p))$, are given by

$$g_i = \dim_k \left(R[T]/(P_1 + \cdots + P_i + L_1 + \cdots + L_{n-i} + L_A + S) \right).$$

Further $g_i = d^i$ for $i < \mathrm{codim}(V(I))$ in $\mathbb{P}^n$.

A New Expression for the Projective Degrees (2)

Theorem 1

- *Where we define the following ideals in $R[T] = k[x_0, \ldots, x_n, T]$:*
  - *Let L_ℓ be an ideal in $R[T]$ defined by a general homogeneous linear form in R for $\ell = 1, \ldots, n$ and let L_A be an ideal in $R[T]$ defined by a general affine linear form in R.*
  - $P_\ell = \left(\sum_{j=0}^{m} \lambda_{\ell,j} f_j \right)$ *for $\lambda_{\ell,j}$ a general scalar in k.*
  - $S = \left(1 - T \cdot \sum_{j=0}^{m} \vartheta_j f_j \right)$, *for ϑ_j a general scalar in k.*

- From this we construct a probabilistic algorithm which can be implemented with symbolic or numeric methods.

Segre Classes and the Projective Degrees

- Consider a subscheme $Y = V(J) \subset \mathbb{P}^n$ defined by a homogeneous ideal $J = (w_0, \ldots, w_m) \subset R = k[x_0, \ldots, x_n]$.
  - Assume, without loss of generality, that $\deg(w_i) = d$ for all i.
  - Let $(g_0, \ldots, g_n)$ be the projective degrees of the rational map $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$, $\phi : p \mapsto (w_0(p) : \cdots : w_m(p))$.

By Proposition 3.1 of Aluffi [2] we have

$$s(Y, \mathbb{P}^n) = 1 - \sum_{i=0}^{n} \frac{g_i h^i}{(1 + dh)^{i+1}} \in A^*(\mathbb{P}^n). \tag{1}$$

This together with Theorem 1 can be used to construct an algorithm to compute Segre classes.

A New Algorithm to Compute the Segre Class

- **Input:** A homogeneous ideal $J = (w_0, \ldots, w_m)$ in $k[x_0, \ldots, x_n]$ defining a scheme $Y = V(J)$ in $\mathbb{P}^n$, without loss of generality assume $\deg(w_i) = d$ for all i.
- **Output:** $s(Y, \mathbb{P}^n) \in A^*(\mathbb{P}^n)$. [*This method is probabilistic.*]
    - Let ϕ be the rational map $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$, $\phi : p \mapsto (w_0(p) : \cdots : w_m(p))$.
    - Compute the projective degrees $(g_0, \ldots, g_n)$ of the rational map ϕ using Theorem 1.
    - Compute $s(Y, \mathbb{P}^n) = 1 - \sum_{i=0}^{n} \frac{g_i h^i}{(1+dh)^{i+1}} \in A^*(\mathbb{P}^n) \cong \mathbb{Z}[h]/(h^{n+1})$.

†† Note that if V is a smooth subscheme of $\mathbb{P}^n$ then we have that

$$c(V) = c(T_V) \cdot [V] = (1 + h)^{n+1} \cdot s(V, \mathbb{P}^n).$$

Segre Class Algorithm Running Times

Input	Segre(Aluffi [2])	segreClass(E.J.P. [8])	**Theorem 1**
Rational normal curve in $\mathbb{P}^7$	-	7s (9s)	0.5s (15s)
Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^3$ in $\mathbb{P}^{11}$	2s	-	3.0s
Smooth deg. 81 variety in $\mathbb{P}^7$	-	36.4s	8.2s
Degree 10 variety in $\mathbb{P}^8$	-	59s	0.9s
Degree 21 variety in $\mathbb{P}^9$	0.5 s	33s	0.9s
Degree 48 variety in $\mathbb{P}^6$	-	173s	2.9s
Codim 2 with 3 generators in $\mathbb{P}^{13}$	-	365.1s	1.3s
Degree 81 variety in $\mathbb{P}^{19}$	-	-	47s
Degree 27 variety in $\mathbb{P}^{15}$	-	-	27s
Degree 12 variety in $\mathbb{P}^{16}$	-	-	4.9s

Table : All algorithms are implemented in Macaulay2 [9]. Timings in () are for the numerical versions, using Bertini [6] (via M2). Computations that do not finish within 600s are denoted -. Numerical timings which did not finish in 600s are omitted. Test computations for the symbolic version were performed over $\mathbb{GF}(32749)$.

Segre Class Algorithm: Complexity Bounds

- Let $\delta(D, N)$ be the arithmetic operations required to find the number of points in a zero dimensional affine variety W defined by N degree D polynomials in N variables.
- Using the algorithm of Lecerf [12] we have that the number of arithmetic operations to solve such a system is polynomial in $\mathcal{O}(D^{3N})$.

Proposition 2

The number of arithmetic operations required to compute the Segre class $s(V, \mathbb{P}^n)$ (for $V = V(f_0, \dots, f_m)$, $\deg(f_i) = d$) using the algorithm described above has order

$$\mathcal{O}(\dim(V) \cdot \delta(d, n+2)).$$

Relating the c_{SM} Class to the Segre Class

- Let M be a smooth variety.
- Theorem I.4 of Alufffi [1] gives the following result for a hypersurface $V = V(f) \subset M$,

$$c_{SM}(V) = c(T_M) \cdot \left(s(V, M) + \sum_{m=0}^{n} \sum_{j=0}^{n-m} \binom{n-m}{j} (-[V])^j \cdot (-1)^{n-m-j} s_{m+j}(Y, M) \right).$$

- Here Y is the singularity subscheme of V i.e. Y is the scheme defined by the vanishing of the partial derivatives of f.
- We can convert this to an expression in terms of the projective degrees of a rational map when working in $\mathbb{P}^n$.

Chern-Schwartz-MacPherson Class of a Hypersurface

- Let $V = V(f) \subset \mathbb{P}^n$ be a hypersurface and let $(g_0, \ldots, g_n)$ be the projective degrees of the polar map $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$

$$\varphi : p \mapsto \left(\frac{\partial f}{\partial x_0}(p) : \cdots : \frac{\partial f}{\partial x_n}(p) \right). \tag{2}$$

- Combining the expression for the Segre class in terms on the projective degree (1) with Theorem I.4 of Alufffi [1] we have:

$$c_{SM}(V) = (1+h)^{n+1} - \sum_{j=0}^{n} g_j (-h)^j (1+h)^{n-j} \text{ in } A^*(\mathbb{P}^n) \cong \mathbb{Z}[h]/(h^{n+1}). \tag{3}$$

- This allows us to compute the c_{SM} class of any projective hypersurface using the projective degrees.

Previous Algorithms to Compute the c_{SM} Class

Consider the hypersurface $V(f) \subset \mathbb{P}^n$ defined by the homogeneous polynomial f.

- All methods employ Theorem I.4 of Alufffi [1], which may be expressed as
$$c_{SM}(V(f)) = (1+h)^{n+1} - \sum_{j=0}^{n} g_j(-h)^j(1+h)^{n-j} \text{ in } A^*(\mathbb{P}^n).$$
- The differences in the methods lay in how the g_j's are understood and computed.
- The first algorithm to compute $c_{SM}(V(f))$ was that of Aluffi [2].
  - To compute the g_j's this algorithm requires the computation of the blowup of $\mathbb{P}^n$ along the singularity subscheme of $V(f)$.
  - The computation of such blowups can be an expensive operation, making this algorithm impractical for many examples.

Previous Algorithms to Compute the c_{SM} Class

- Another algorithm to compute the c_{SM} class of a hypersurface was given by Jost in [11].
  - This method is probabilistic and finds the g_j's by computing the degrees of certain residual sets via a particular saturation.
- In the algorithm we will discuss here the g_i's, are understood to be the projective degree of a rational map defined by the polynomials $(\frac{df}{dx_0}, \dots, \frac{df}{dx_n})$. This preservative leads to a new probabilistic method to compute $c_{SM}(V(f))$ and provides a performance improvement in many cases.
- All these methods require the use of the inclusion-exclusion property of c_{SM} classes when the scheme $V \subset \mathbb{P}^n$ has codimension higher than one.

Inclusion/Exclusion for c_{SM} Classes

- Specifically for V_1, V_2 subschemes of $\mathbb{P}^n$ the inclusion-exclusion property for c_{SM} classes states

$$c_{SM}(V_1 \cap V_2) = c_{SM}(V_1) + c_{SM}(V_2) - c_{SM}(V_1 \cup V_2). \quad (4)$$

- Inclusion/Exclusion allows for the computation of $c_{SM}(V)$ for V of any codimension.
- This requires exponentially many c_{SM} computations relative to the number of generators of I.
- Must consider c_{SM} classes of products of many or all of the generators of I, which may have significantly higher degree than the original scheme V.

A New Algorithm to Compute the Chern-Schwartz-MacPherson Class

- **Input:** A homogeneous ideal $(f_0, \ldots, f_r)$ in $k[x_0, \ldots, x_n]$ defining a scheme $V = V(I)$ in $\mathbb{P}^n$.
- **Output:** $c_{SM}(V)$ and/or $\chi(V)$. [*This method is probabilistic.*]
  - Make a list $\mathcal{L}_I$ of all generators and all products of generators of the ideal I.
  - For each polynomial $f \in \mathcal{L}_I$ compute $c_{SM}(V(f))$.
    - Compute the projective degrees $(g_0, \ldots, g_n)$ of the polar map of f (2) using Theorem 1.
    - Compute $c_{SM}(V(f))$ from the projective degree $(g_0, \ldots, g_n)$ using (21).
  - Apply the inclusion/exclusion property of c_{SM} classes to obtain $c_{SM}(V)$.

Running times for c_{SM} Algorithm

Input	CSM (Alu.)	CSM (Jost)	**csm_pol (M2)**	**csm_pol (Sage)**	euler
Twisted cubic	0.3s	0.1s (35s)	0.1s (37s)	0.1s (0.6s)	0.2s
Seg. embed. $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$	0.4s	0.8s (148s)	0.7s (152s)	0.2s (57s)	0.2s
Smooth degree 8 in $\mathbb{P}^4$	-	1.2s (-)	0.5s (-)	0.2s (28s)	20.1s
Smooth degree 4 in $\mathbb{P}^{10}$	-	56.8s	2.3s	2.2s	-
Smooth degree 6 in $\mathbb{P}^7$	-	-	10.5s	77.7s	-
Deg. 12 codim. 1 in $\mathbb{P}^3$	25.3s	1.0s	0.1s	0.1s	n/a
Degree 3 variety in $\mathbb{P}^8$	-	85.2s	4.7s	1.0s	n/a
Degree 16 variety in $\mathbb{P}^{10}$	-	-	0.6s	2.3s	n/a
Degree 16 variety in $\mathbb{P}^5$	-	-	0.6s	0.3s	n/a

Table : Timings in () are for the numerical versions, using either Bertini [6] (via M2) or PHCpack [15] (via Sage). Computations that do not finish within 600s are denoted -, or are omitted for numerical versions. Test computations for the symbolic version were performed over $\mathbb{GF}(32749)$

c_{SM} Class Algorithm: Complexity Bounds

- As before $\delta(D, N)$ is the arithmetic operations required to find the number of points in a zero dimensional affine variety W defined by N degree D polynomials in N variables.
- Using the algorithm of Lecerf [12] we have that the number of arithmetic operations to solve such a system is polynomial in $\mathcal{O}(D^{3N})$.

Proposition 3

The number of arithmetic operations required to compute $c_{SM}(V)$, $V = V(f_0, \ldots, f_m)$, using the algorithm described above has order

$$\mathcal{O}\left(2^{m+1} n \cdot \delta((m+1) \cdot d, n+2)\right).$$

Toric Case: Setting and Notation

- Let X_Σ be an n dimensional smooth complete toric variety defined by a fan Σ.
- Let R be the graded coordinate ring (Cox ring) of X_Σ with irrelevant ideal B and assume that all Cartier divisors D_{ρ_j} associated to generating rays ρ_j in $\Sigma(1) = \{\rho_1, \dots, \rho_m\}$ are nef (numerically effective).
- These assumptions are satisfied by $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\tau}$, for example.
- We work in the Chow ring of X_Σ.
- When $X_\Sigma = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\tau}$ the Chow ring is

$$A^*(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\tau}) \cong \mathbb{Z}[h_1, \dots, h_\tau]/(h_1^{n_1+1}, \dots, h_\tau^{n_\tau+1}),$$

which h_j the rational equivalence class of a general hyperplane in $\mathbb{P}^j$

Chow Ring of a Smooth Complete Toric Variety

- Let $x_1 = x_{\rho_1}, \ldots, x_m = x_{\rho_m}$ be the intermediates of R.
- We express the Chow ring as

$$A^*(X_\Sigma) \cong \mathbb{Z}[x_1, \ldots, x_m]/(\mathcal{I} + \mathcal{J})$$

  with $[V(\rho_i)] \cong x_i$.
- $\mathcal{I}$ is the Stanley-Reisner ideal of the fan Σ, i.e. $\mathcal{I} = (x_{i_1} \cdots x_{i_s} \mid i_{i_j} \text{ distinct and } \rho_{i_1} + \cdots + \rho_{i_s} \text{ is not a cone of } \Sigma)$.
- $\mathcal{J}$ denotes the ideal of $\mathbb{Z}[x_1, \ldots, x_m]$ generated by linear relations of the rays $\Sigma(1) = \{\rho_1, \ldots, \rho_m\}$.

Projective Degrees for Toric Varieties

- Let I be an ideal in R which is homogeneous with respect to the grading, we may choose generators $I = (f_0, \ldots, f_r)$ so that $[V(f_i)] = \alpha \in A^1(X_\Sigma)$ for all i.
- Let $V = V(I)$ be the closed subscheme of X_Σ defined by I.
- Define a rational map $\phi : X_\Sigma \dashrightarrow \mathbb{P}^r$ given by

$$\phi : p \mapsto (f_0(p) : \cdots : f_r(p)). \tag{5}$$

- Let $[Y_\iota] = \left[\phi^{-1}(\mathbb{P}^{r-\iota})\right] \in A^*(X_\Sigma)$ where $\mathbb{P}^{r-\iota}$ denotes a general hyperplane of dimension $r - \iota$ in $\mathbb{P}^r$.

Projective Degrees for Toric Varieties

- $[Y_\iota] \in A^\iota(X_\Sigma)$, that is the cycle $[Y_\iota]$ has pure codimension ι.
- Define the class

$$G = \sum_{\iota=0}^{n} [Y_\iota] = \sum_{i}^{\mu} \gamma_i^{(\iota)} \omega_i^{(\iota)} \in A^*(X_\Sigma), \tag{6}$$

  where $\omega_1^{(\iota)}, \ldots, \omega_\mu^{(\iota)}$ is a basis of $A^\iota(X_\Sigma)$.
- We refer to the $\gamma_i^{(\iota)}$ as the *projective degrees* of the rational map ϕ.
- One can show that $[Y_\iota] = \alpha^\iota$ for $\iota < \mathrm{codim}(V)$ since V has no components of codimension less than $\mathrm{codim}(V)$.

A New Expression for $s(V, X_\Sigma)$ via Projective Degrees for Toric Varieties

Theorem 4

Let $I = (f_0, \ldots, f_r)$ be an ideal homogeneous with respect to the grading on $R = k[x_1, \ldots, x_m]$. Consider the scheme $V = V(I)$, and assume, without loss of generality, that $[V(f_i)] = \alpha \in A^1(X_\Sigma)$ for all i. With G as in (6) we have

$$s(V, X_\Sigma) = 1 - \frac{1}{(1+\alpha)} \left(\sum_{i \geq 0} \frac{G^{(i)}}{(1+\alpha)^i} \right).$$

Here $G^{(i)}$ is the codimension i piece of G.

- This gives the total Chern class (for smooth V)

$$c(V) = c(T_V) \cdot [V] = (1+x_1) \cdots (1+x_m) s(V, X_\Sigma).$$

A New Expression for the Projective Degrees (1)

Theorem 5

Let $I = (f_0, \ldots, f_r)$ be an ideal of R homogeneous w.r.t. to the grading, $V = V(I) \subset X_\Sigma$. $[V(f_j)] = \alpha$ for all j. Then we have that the projective degrees are given by

$$\gamma_i^{(\iota)} = \dim_k \left(R[T]/(P_1 + \cdots + P_\iota + L_{a_i^{(\iota)}} + L_A + S) \right),$$

where $[Y_\iota] = \sum_i^\mu \gamma_i^{(\iota)} \omega_i^{(\iota)}$ and

$$G = \left(1 + \sum_{\iota=1}^{\mathrm{codim}(V)-1} \alpha^\iota + \sum_{\iota=\mathrm{codim}(V)}^{n} [Y_\iota] \right)$$ *in the Chow ring $A^*(X_\Sigma)$.*

- *From this we construct a probabilistic algorithm which can be implemented with symbolic or numeric methods.*

A New Expression for the Projective Degrees (2)

Theorem 5

- *General linear combinations* $P_1 + \cdots + P_\iota$*:*
  - $P_j = \sum_{l=0}^{r} \lambda_{j,l} f_l$ *for* $j = 1, \ldots, n$ *and for general* $\lambda_{j,l}$.
- *General linear forms* $L_{a_i^{(\iota)}}$*:*
  - *We are finding the coefficient of the basis element* $\omega_i^{(\iota)} \in A^\iota(X_\Sigma)$.
  - *Let* $[V(\sigma)] = [V(\rho_1 + \cdots + \rho_m)]$ *be the basis of* $A_0(X_\Sigma)$.
  - *Set* $a_i^{(\iota)} = \frac{[V(\sigma)]}{\omega_i^{(\iota)}} \in A^*(X_\Sigma)$.
  - *Factor* $a_i^{(\iota)}$ *so that* $a_i^{(\iota)} = b_1^{j_1} \cdots b_q^{j_q}$ *for* $b_1, \ldots, b_q \in A^1(X_\Sigma)$.
  - *Let* $\ell(b) \in R$ *be general with* $[\ell(b)] = b \in A^*(X_\Sigma)$ *for* $b \in A^1(X_\Sigma)$.
  - *Let* $L_{a_i^{(\iota)}}$ *be the ideal generated by* j_1 *linear forms* $\ell(b_1)$, j_2 *linear forms* $\ell(b_2), \ldots,$ *and* j_q *linear forms* $\ell(b_q)$.

A New Expression for the Projective Degrees (3)

Theorem 5

- *General affine linear forms (to dehomogenize):*
  - $\mathfrak{p}_1, \ldots, \mathfrak{p}_\upsilon$ *be the primary ideals in primary decomposition of* B*, further assume that* $\mathfrak{p}_l = (x_{s_1(l)}, \ldots, x_{s_j(l)})$.
  - $L_A = \left(\sum_{j=1}^{\nu_1} \lambda_j^{(1)} x_{s_j(1)} - 1, \ldots, \sum_{j=1}^{\nu_\upsilon} \lambda_j^{(\upsilon)} x_{s_j(\upsilon)} - 1 \right)$*, for general* $\lambda_j^{(l)}$.
- *Removing* $V(I)$*:*
  - S *is an ideal given by* $S = \left(1 - T \sum_{l=0}^{r} \vartheta_l f_l \right)$ *for general* ϑ_l.

Performance

The Segre Class Algorithm for Subschemes of X_Σ

- We compare the run times of our algorithm (SegreMultiProj) to the run times of the algorithm of Moe and Qviller [13].
- Computations performed in M2 (V1.7) [9] over $\mathbb{GF}(32749)$.

Input	toricSegreClass ([13])	SegreProjectiveDegee
Codimension 3 in $\mathbb{P}^2 \times \mathbb{P}^3$	-	33.6s
Codimension 2 in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	32.0s	0.1s
Hypersurface in $\mathbb{P}^5 \times \mathbb{P}^3$	147.4s	0.5s
Codimension 2 in $\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^1$	66.8s	0.5s
Codimension 2 in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$	15.7s	0.5s
Codimension 2 in $\mathbb{P}^4 \times \mathbb{P}^3 \times \mathbb{P}^3$	-	7.4s
Codimension 2 in $\mathbb{P}^4 \times \mathbb{P}^3 \times \mathbb{P}^5$	-	22.4s
Codimension 4 in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$	-	2.7s
Codim. 1 with 2 gens. in Dim. 3 X_{Σ_1}	7.6s	0.1s
Codim. 1 with 3 gens. in Dim. 3 X_{Σ_1}	-	1.0s

Performance

$c_{SM}(V)$ and/or $\chi(V)$ for V a Subscheme of some X_Σ

- This is the first algorithm known to us in this setting.
  - For the last example around 85% of the time is spent computing the c_{SM} class of the ideal given by the product of all 3 generators.
  - To compute this class we must solve 35 zero dimensional systems in 11 variables which have 2, 3, 3, 3, 6, 6, 6, 9, 6, 4, 9, 9, 12, 18, 12, 18, 18, 12, 12, 18, 27, 18, 36, 36, 36, 24, 36, 24, 54, 54, 72, 72, 72, 108, and 144 solutions, respectively.

Input	c_{SM} (Projective Degree)
Codimension 1 in with 2 gens. Dim. 3 X_{Σ_1}	0.3s
Codimension 1 in with 3 gens. Dim. 3 X_{Σ_1}	2.0s
Codim. 2 in $\mathbb{P}^2 \times \mathbb{P}^2$	0.3s
Codim. 2 in $\mathbb{P}^6 \times \mathbb{P}^2$ with degree $(3,0),(0,2)$ eqs.	3.9s
Codim. 2 in $\mathbb{P}^5 \times \mathbb{P}^3$ with degree $(2,1)$ and $(1,1)$ eqs.	12.4s
Codim. 2 in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3$ with degree $(2,1,0)$ and $(0,1,2)$ eqs.	4.8s
Codim. 3 in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3$ with degree $(2,1,0),(0,1,2),(1,2,0)$ eqs.	52.4s

The c_{SM} class of a Complete Simplicial Toric Variety

- Consider a complete simplicial toric variety X_Σ defined by a fan Σ.
- We give a combinatorial method to compute the class $c_{SM}(X_\Sigma)$ in the rational Chow ring, $A^*(X_\Sigma)_\mathbb{Q}$ of X_Σ.
- The Chow ring of a simplicial toric variety has the same structure as that of a smooth toric variety, except rather than working over the integers the coefficient ring is the ring of rational numbers $\mathbb{Q}$.
- For a cone $\sigma = \rho_1 + \cdots + \rho_d \in \Sigma$ let $\mathfrak{M}_\sigma$ be the matrix with columns specified by the generating vectors of the rays $\rho_1, \ldots, \rho_d$.
- Let Herm denote the Hermite normal form of a matrix.
- One may show that $\mathrm{mult}(\sigma) = |\det(\mathrm{Herm}(\mathfrak{M}_\sigma))|$.
- If X_Σ is smooth $\mathrm{mult}(\sigma) = 1$ for all $\sigma \in \Sigma$.

The c_{SM} class of a Complete Simplicial Toric Variety

Our Algorithm

- **Input:** A Complete Simplicial Toric Variety X_Σ, $\dim(X_\Sigma) = n$.
- **Output:** $c_{SM}(X_\Sigma)$.
  - Denote the set of generating rays as $\Sigma(1) = \{\rho_1, \dots \rho_m\}$.
  - **For each codimension** $i = 1$ **to** n **do:**
    - Compute the codimension i piece of $c_{SM}(X_\Sigma)$:

$$c_{SM}(X_\Sigma)^{(i)} = \sum_{\substack{\sigma = \rho_{j_1} + \cdots + \rho_{j_i} \\ \text{for every } \{\rho_{j_1}, \dots, \rho_{j_i}\} \subset \Sigma(1) \\ \text{having } i \text{ elements}}} [V(\sigma)] \in A^*(X_\Sigma)_{\mathbb{Q}},$$

$$[V(\sigma)] = \begin{cases} 0 \text{ if } \sigma \notin \Sigma \\ \text{mult}(\sigma) x_{j_1} \cdots x_{j_i} = |\det(\text{Herm}(\mathfrak{M}_\sigma))| \, x_{j_1} \cdots x_{j_i} \text{ otherwise} \end{cases}.$$

  - **Return** $c_{SM}(X_\Sigma) = 1 + \sum_{i=1}^{n} c_{SM}(X_\Sigma)^{(i)} \in A^*(X_\Sigma)_{\mathbb{Q}}$.

The c_{SM} class of a Complete Simplicial Toric Variety

- The † denotes that for these versions of the algorithms the input is checked for smoothness.
- If the input is found to be smooth we know $\mathrm{mult}(\sigma) = 1$ for all cones $\sigma \in \Sigma$ and hence we do not compute Hermite normal forms and determinates.

Input	c_{SM} Alg. †	c_{SM} Alg.	Chow Ring
$\mathbb{P}^6$	0.0s	0.0s	0.1 s
$\mathbb{P}^{12}$	0.2s	3.8s	0.3 s
$\mathbb{P}^{16}$	5.3s	85.4s	0.7 s
$\mathbb{P}^5 \times \mathbb{P}^6$	0.3s	3.7s	1.2 s
$\mathbb{P}^5 \times \mathbb{P}^8$	1.1s	16.8s	2.1 s
$\mathbb{P}^8 \times \mathbb{P}^8$	12.0s	168.5s	4.5 s
$\mathbb{P}^5 \times \mathbb{P}^5 \times \mathbb{P}^5$	12.8s	156.7s	11.8 s
$\mathbb{P}^5 \times \mathbb{P}^5 \times \mathbb{P}^6$	28.4s	387.1s	17.0 s
Fano sixfold 123	0.3s	1.0s	1.1 s
Fano sixfold 1007	0.4s	1.0s	1.8 s

Future/Current Work & Demo

- Try using the algorithm on examples arising from applications and see how it performs there.
- Investigate possible improvements to the M2 implementation of the algorithm.
- Extend the algorithms presented here to compute the c_{SM} class and Segre class of a subscheme of any smooth complete toric variety.
- Investigate whether one may consider the c_{SM} class as a more refined version of the Euler characteristic in a precise sense for subschemes of more general varieties (i.e. toric varieties).
- Code avalibe as part of the "CharacteristicsClasses" M2 package, on github:

  `https://github.com/Macaulay2/M2/blob/master/M2/Macaulay2/packages/CharacteristicClasses.m2.`
- Will be included in the next M2 release.

Thank you for listening!

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