Algorithms to Compute Chern-Schwartz-Macpherson and Segre Classes and the Euler Characteristic

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Overview

We discuss algorithms to compute some topological invariants (in particular characteristics classes of schemes) for subschemes of certain smooth complete toric varieties X_Σ. We work over k, an algebraicly closed feild of chacteristic zero.

- We consider probabilistic algorithms to compute the Euler characteristic and the Chern-Schwartz-MacPherson and Segre class of a subscheme V of X_Σ.
- All these methods can be (and are) implemented symbolically using Gröbner bases calculations, or numerically using homotopy continuation.
- The algorithms are tested on several examples and are found to perform favourably compared to other existing algorithms.

The topological Euler characteristic

Applications and Computation

- The Euler characteristic χ is an important topological invariant.
- Some recent applications include:
  - Maximum likelihood estimation in algebraic statistics by Huh [12] and by Rodriguez and Wang [14].
  - Applications to string theory in physics by Collinucci et al. [8] and by Aluffi and Esole [5].
- For projective schemes it can be computed several different ways, for example from Hodge numbers, or as we do here, from the Chern-Schwartz-Macperhson class.
- In particular from $c_{SM}(V)$ we may immediately obtain the Euler characteristic of V, since $\chi(V)$ is equal to the degree of the zero dimensional component of $c_{SM}(V)$.

Chern-Schwartz-MacPherson Classes

- The c_{SM} class generalizes the Chern class of the tangent bundle to singular varieties/schemes, i.e. $c(T_V) \cdot [V] = c_{SM}(V)$ when V is a smooth subscheme of a smooth variety M.
- The c_{SM} class has important functorial properties, and relations to the Euler characteristic.
- The Chern-Schwartz-MacPherson class has also been directly related to problems in string theory in Aluffi and Esole [4].
- For subschemes of $\mathbb{P}^n$ the relationship between the c_{SM} class and the Euler characteristic gives us a more concrete way to understand the information contained in the c_{SM} class.

Chern-Schwartz-MacPherson Classes

- When V is a subscheme of $\mathbb{P}^n$ the class $c_{SM}(V)$ can be thought of as a more refined version of the Euler characteristic since it contains the Euler characteristics of V and those of general linear sections of V for each codimension.
- Specifically (see Aluffi [3]), if $\dim(V) = m$, starting from $c_{SM}(V)$ we may directly obtain the list of invariants

$$\chi(V), \chi(V \cap L_1), \chi(V \cap L_1 \cap L_2), \ldots, \chi(V \cap L_1 \cap \cdots \cap L_m)$$

  where $L_1, \ldots, L_m$ are general hyperplanes.
- Alternatively one may obtain $c_{SM}(V)$ from the list of Euler characteristics above as well, i.e. there is an involution.

Segre Class of a Subscheme

- For V a proper closed subscheme of a variety W the Segre class $s(V, W)$ is the class of a particular vector bundle (specifically the normal cone $C_V W$).
- Can also write $s(V, W) = \eta_* \left(\frac{[\tilde{V}]}{1 + [\tilde{V}]} \right) \in A^*(V)$, where $\tilde{V}$ is the exceptional divisor of $Bl_V W$, the blowup of W along V and $\eta : \tilde{V} \to V$ is the projection.
- Can be used to define several other characteristic classes in some settings (e.g. Chern, Chern-Fulton, c_{SM}, and others).
- For subschemes of the smooth complete toric varieties we consider here we will give a more explicit (and easier to compute) expression in terms of the projective degrees below.

Setting and Notation

- Let X_Σ be an n dimensional smooth complete toric variety defined by a fan Σ.
- Let R be the graded coordinate ring (Cox ring) of X_Σ with irrelevant ideal B and assume that all Cartier divisors D_{ρ_j} associated to generating rays ρ_j in $\Sigma(1) = \{\rho_1, \ldots, \rho_m\}$ are nef.
- We also require an additional technical assumption.
- These assumptions are satisfied by $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\tau}$, for example.
- We work in the Chow ring of X_Σ.
- When $X_\Sigma = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\tau}$ the Chow ring is

$$A^*(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\tau}) \cong \mathbb{Z}[h_1, \ldots, h_\tau]/(h_1^{n_1+1}, \ldots, h_\tau^{n_\tau+1}),$$

which h_j the rational equivalence class of a general hyperplane in $\mathbb{P}^j$

Chow Ring of a Smooth Complete Toric Variety

- Let $x_1 = x_{\rho_1}, \ldots, x_m = x_{\rho_m}$ be the intermediates of R.
- We express the Chow ring as

$$A^*(X_\Sigma) \cong \mathbb{Z}[x_1, \ldots, x_m]/(\mathcal{I} + \mathcal{J})$$

  with $[V(\rho_i)] \cong x_i$.
- $\mathcal{I}$ is the Stanley-Reisner ideal of the fan Σ, i.e.
  $\mathcal{I} = (x_{i_1} \cdots x_{i_s} \mid i_{i_j} \text{ distinct and } \rho_{i_1} + \cdots + \rho_{i_s} \text{ is not a cone of } \Sigma)$.
- $\mathcal{J}$ denotes the ideal of $\mathbb{Z}[x_1, \ldots, x_m]$ generated by linear relations of the rays $\Sigma(1) = \{\rho_1, \ldots, \rho_m\}$.

Previous Algorithms to Compute Segre Classes in $\mathbb{P}^n$

- The first algorithm to compute the Segre class $s(V, \mathbb{P}^n)$ for $V = V(I)$ a subscheme of $\mathbb{P}^n$ was that of Aluffi [2].
  - The algorithm of Aluffi works by computing the blowup $Bl_V\mathbb{P}^n$ of $\mathbb{P}^n$ along V (or equivalently by computing the Rees algebra of I).
  - This is a quite expensive computation in most cases.
- Another algorithm to compute $s(V, \mathbb{P}^n)$ is that of Eklund, Jost and Peterson [9].
  - This method is probabilistic and computes the Segre class by computing the degrees of certain residual sets via a particular saturation.
- In [11] we gave an algorithm to compute $s(V, \mathbb{P}^n)$ via projective degree computation; the main step is finding the vector space dimension of a ring modulo a zero dimensional ideal.

Previous Algorithms to compute $s(V, X_\Sigma)$

- The only previous algorithm known to us which is applicable to compute the Segre class of a subscheme of X_Σ (with X_Σ having the form above) is that of Moe and Qviller [13].
- This algorithm is subject to the same nefness restriction on the divisors as the algorithm presented here.
- Its main computational step is finding residual sets by saturation and computing their multi-degrees.
- Generalizes the algorithm of Eklund, Jost, and Peterson[9].
- We now describe an algorithm which generalizes our projective degree method for $\mathbb{P}^n$ to the case when V is a subscheme of X_Σ.

Projective Degrees

- Let I be an ideal in R which is homogeneous with respect to the grading, we may choose generators $I = (f_0, \ldots, f_r)$ so that $[V(f_i)] = \alpha \in A^1(X_\Sigma)$ for all i.
- Let $V = V(I)$ be the closed subscheme of X_Σ defined by I.
- Define a rational map $\phi : X_\Sigma \dashrightarrow \mathbb{P}^r$ given by

$$\phi : p \mapsto (f_0(p) : \cdots : f_r(p)). \tag{1}$$

- Let $[Y_\iota] = \left[\phi^{-1}(\mathbb{P}^{r-\iota})\right] \in A^*(X_\Sigma)$ where $\mathbb{P}^{r-\iota}$ denotes a general hyperplane of dimension $r - \iota$ in $\mathbb{P}^r$.

Projective Degrees

- $[Y_\iota] \in A^\iota(X_\Sigma)$, that is the cycle $[Y_\iota]$ has pure codimension ι.
- Define the class

$$G = \sum_{\iota=0}^{n} [Y_\iota] = \sum_{i}^{\mu} \gamma_i^{(\iota)} \omega_i^{(\iota)} \in A^*(X_\Sigma), \tag{2}$$

  where $\omega_1^{(\iota)}, \dots, \omega_\mu^{(\iota)}$ is a basis of $A^\iota(X_\Sigma)$.
- We refer to the $\gamma_i^{(\iota)}$ as the *projective degrees* of the rational map ϕ.
- One can show that $[Y_\iota] = \alpha^\iota$ for $\iota < \mathrm{codim}(V)$ since V has no components of codimension less than $\mathrm{codim}(V)$.

Projective Degrees

- When $X_{\Sigma} = \mathbb{P}^n$ we have that

$$G = \sum_{\iota=0}^{n} [Y_\iota] = \sum_{\iota=0}^{n} g_\iota h^\iota \in A^*(\mathbb{P}^n) \cong \mathbb{Z}[h]/(h^{n+1}). \qquad (3)$$

- Here $(g_0, \ldots, g_n) \in \mathbb{Z}^{n+1}$ with

$$g_\iota = \mathrm{card}\left(\phi^{-1}\left(\mathbb{P}^{m-\iota}\right) \cap \mathbb{P}^\iota\right),$$

$\mathbb{P}^{m-\iota} \subset \mathbb{P}^m$ and $\mathbb{P}^\iota \subset \mathbb{P}^n$ are general hyperplanes of dimension $m - \iota$ and ι respectively.

- If $\iota \leq \mathrm{codim}(V)$ then $g_\iota = d^\iota$.

A New Expression for $s(V, X_\Sigma)$ via Projective Degrees

Theorem 1

Let $I = (f_0, \ldots, f_r)$ be an ideal homogeneous with respect to the grading on R. Consider the scheme $V = V(I)$, and assume, without loss of generality, that $[V(f_i)] = \alpha \in A^1(X_\Sigma)$ for all i. With G as in (2) we have

$$s(V, X_\Sigma) = 1 - \frac{1}{(1+\alpha)} \left(\sum_{i \geq 0} \frac{G^{(i)}}{(1+\alpha)^i} \right).$$

Here $G^{(i)}$ is the codimension i piece of G.

- In $\mathbb{P}^n$ this simplifies to $\alpha = dh$, $s(V, \mathbb{P}^n) = 1 - \sum_{i=0}^{n} \frac{g_i h^i}{(1+dh)^{i+1}}$.

A New Expression for the Projective Degrees (1)

Theorem 2

Let $I = (f_0, \ldots, f_r)$ be an ideal of R homogeneous w.r.t. to the grading, $V = V(I) \subset X_\Sigma$. $[V(f_j)] = \alpha$ for all j. Then we have that the projective degrees are given by

$$\gamma_i^{(\iota)} = \dim_k \left(R[T]/(P_1 + \cdots + P_\iota + L_{a_i^{(\iota)}} + L_A + S) \right),$$

where $[Y_\iota] = \sum_i^\mu \gamma_i^{(\iota)} \omega_i^{(\iota)}$ *and*

$$G = \left(1 + \sum_{\iota=1}^{\mathrm{codim}(V)-1} \alpha^\iota + \sum_{\iota=\mathrm{codim}(V)}^{n} [Y_\iota] \right)$$ *in the Chow ring $A^*(X_\Sigma)$.*

- *From this we construct a probabilistic algorithm which can be implemented with symbolic or numeric methods.*

A New Expression for the Projective Degrees (2)

Theorem 2

- *General linear combinations* $P_1 + \cdots + P_\iota$:
  - $P_j = \sum_{l=0}^{r} \lambda_{j,l} f_l$ *for* $j = 1, \ldots, n$ *and for general* $\lambda_{j,l}$.
- *General linear forms* $L_{a_i^{(\iota)}}$:
  - *We are finding the coefficient of the basis element* $\omega_i^{(\iota)} \in A^\iota(X_\Sigma)$.
  - *Let* $[V(\sigma)] = [V(\rho_1 + \cdots + \rho_m)]$ *be the basis of* $A_0(X_\Sigma)$.
  - *Set* $a_i^{(\iota)} = \frac{[V(\sigma)]}{\omega_i^{(\iota)}} \in A^*(X_\Sigma)$.
  - *Factor* $a_i^{(\iota)}$ *so that* $a_i^{(\iota)} = b_1^{j_1} \cdots b_q^{j_q}$ *for* $b_1, \ldots, b_q \in A^1(X_\Sigma)$.
  - *Let* $\ell(b) \in R$ *be general with* $[\ell(b)] = b \in A^*(X_\Sigma)$ *for* $b \in A^1(X_\Sigma)$.
  - *Let* $L_{a_i^{(\iota)}}$ *be the ideal generated by* j_1 *linear forms* $\ell(b_1)$, j_2 *linear forms* $\ell(b_2)$,…, *and* j_q *linear forms* $\ell(b_q)$.

A New Expression for the Projective Degrees (3)

Theorem 2

- *General affine linear forms (to dehomogenize):*
  - $\mathfrak{p}_1, \ldots, \mathfrak{p}_\upsilon$ *be the primary ideals in primary decomposition of B, further assume that* $\mathfrak{p}_l = (x_{s_1(l)}, \ldots, x_{s_j(l)})$.
  - $L_A = \left(\sum_{j=1}^{\nu_1} \lambda_j^{(1)} x_{s_j(1)} - 1, \ldots, \sum_{j=1}^{\nu_\upsilon} \lambda_j^{(\upsilon)} x_{s_j(\upsilon)} - 1 \right)$, *for general* $\lambda_j^{(l)}$.
- *Removing* $V(I)$*:*
  - *S is an ideal given by* $S = \left(1 - T \sum_{l=0}^{r} \vartheta_l f_l \right)$ *for general* ϑ_l.

Segre Class Algorithm

Input: X_Σ and an ideal $I = (f_0, \ldots, f_r)$ homogeneous with respect to the grading on the coordinate ring R of X_Σ defining a subscheme $V = V(I)$ of X_Σ. Further assume, without loss of generality, that $[V(f_j)] = \alpha \in A^*(X_\Sigma)$ for all j.
Output: $s(V, X_\Sigma)$ in $A^*(X_\Sigma)$.

- Apply Theorem 2 to compute the projective degrees and construct the class $G = \left(1 + \sum_{\iota=1}^{\mathrm{codim}(V)-1} \alpha^\iota + \sum_{\iota=\mathrm{codim}(V)}^{n} [Y_\iota]\right) \in A^*(X_\Sigma)$
- Apply Theorem 1 to construct the Segre class

$$s(V, X_\Sigma) = 1 - \frac{1}{(1+\alpha)}\left(\sum_{i \geq 0} \frac{G^{(i)}}{(1+\alpha)^i}\right).$$

Performance

The Segre Class Algorithm for Subschemes of $\mathbb{P}^n$

Input	Segre(Aluffi [2])	segreClass(E.J.P. [9])	**Theorem 1**
Rational normal curve in $\mathbb{P}^7$	-	7s (9s)	0.5s (15s)
Smooth deg. 81 variety in $\mathbb{P}^7$	-	36.4s	8.2s
Degree 10 variety in $\mathbb{P}^8$	-	59s	0.9s
Degree 21 variety in $\mathbb{P}^9$	0.5 s	33s	0.9s
Degree 48 variety in $\mathbb{P}^6$	-	173s	2.9s
Codim 2 with 3 generators in $\mathbb{P}^{13}$	-	365.1s	1.3s
Degree 81 variety in $\mathbb{P}^{19}$	-	-	47s
Degree 27 variety in $\mathbb{P}^{15}$	-	-	27s
Degree 12 variety in $\mathbb{P}^{16}$	-	-	4.9s

Table : All algorithms are implemented in Macaulay2 [10]. Timings in () are for the numerical versions, using Bertini [6] (via M2). Computations that do not finish within 600s are denoted -. Numerical timings which did not finish in 600s are omitted. Test using symbolic version performed over $\mathbb{GF}(32749)$.

Performance

The Segre Class Algorithm for Subschemes of X_Σ

- We compare the run times of our algorithm (SegreMultiProj) to the run times of the algorithm of Moe and Qviller [13].
- Computations performed in M2 (V1.7) [10] over $\mathbb{GF}(32749)$.

Input	toricSegreClass ([13])	SegreProjectiveDegee
Codimension 3 in $\mathbb{P}^2 \times \mathbb{P}^3$	-	33.6s
Codimension 2 in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	32.0s	0.1s
Hypersurface in $\mathbb{P}^5 \times \mathbb{P}^3$	147.4s	0.5s
Codimension 2 in $\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^1$	66.8s	0.5s
Codimension 2 in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$	15.7s	0.5s
Codimension 2 in $\mathbb{P}^4 \times \mathbb{P}^3 \times \mathbb{P}^3$	-	7.4s
Codimension 2 in $\mathbb{P}^4 \times \mathbb{P}^3 \times \mathbb{P}^5$	-	22.4s
Codimension 4 in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$	-	2.7s
Codim. 1 with 2 gens. in Dim. 3 X_{Σ_1}	7.6s	0.1s
Codim. 1 with 3 gens. in Dim. 3 X_{Σ_1}	-	1.0s

Relating the c_{SM} Class to the Segre Class

- Let M be a smooth variety.
- Theorem I.4 of Alufffi [1] gives the following result for a hypersurface $V = V(f) \subset M$,

$$c_{SM}(V) = c(T_M) \cdot \left(s(V,M) + \sum_{m=0}^{n} \sum_{j=0}^{n-m} \binom{n-m}{j} (-[V])^j \cdot (-1)^{n-m-j} s_{m+j}(Y,M) \right).$$

- Here Y is the singularity subscheme of V i.e. Y is the scheme defined by the vanishing of the partial derivatives of f.
- We can convert this to an expression in terms of the projective degrees via our Segre class formula......

Inclusion/Exclusion for c_{SM} Classes

- Specifically for V_1, V_2 subschemes of $\mathbb{P}^n$ the inclusion-exclusion property for c_{SM} classes states

$$c_{SM}(V_1 \cap V_2) = c_{SM}(V_1) + c_{SM}(V_2) - c_{SM}(V_1 \cup V_2). \quad (4)$$

- Inclusion/Exclusion allows for the computation of $c_{SM}(V)$ for V of any codimension.
- This requires exponentially many c_{SM} computations relative to the number of generators of I.
- Must consider c_{SM} classes of products of many or all of the generators of I, which may have significantly higher degree than the original scheme V.

Performance

$c_{SM}(V)$ and/or $\chi(V)$ for V a Subscheme of $\mathbb{P}^n$

Input	CSM (Alu.)	CSM (Jost)	**csm_pol (M2)**	euler
Twisted cubic	0.3s	0.1s (35s)	0.1s (37s)	0.2s
Seg. embed. $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$	0.4s	0.8s (148s)	0.7s (152s)	0.2s
Smooth degree 8 in $\mathbb{P}^4$	-	1.2s (-)	0.5s (-)	20.1s
Smooth degree 4 in $\mathbb{P}^{10}$	-	56.8s	2.3s	-
Smooth degree 6 in $\mathbb{P}^7$	-	-	10.5s	-
Deg. 12 codim. 1 in $\mathbb{P}^3$	25.3s	1.0s	0.1s	n/a
Degree 3 variety in $\mathbb{P}^8$	-	85.2s	4.7s	n/a
Degree 16 variety in $\mathbb{P}^{10}$	-	-	0.6s	n/a
Degree 16 variety in $\mathbb{P}^5$	-	-	0.6s	n/a

Table : Timings in () are for the numerical versions, using either Bertini [6] (via M2 [10]). Computations that do not finish within 600s are denoted -, or are omitted for numerical versions. Test computations for the symbolic version were performed over $\mathbb{GF}(32749)$

Performance

$c_{SM}(V)$ and/or $\chi(V)$ for V a Subscheme of some X_Σ

- This is the first algorithm known to us in this setting.
  - For the last example around 85% of the time is spent computing the c_{SM} class of the ideal given by the product of all 3 generators.
  - To compute this class we must solve 35 zero dimensional systems in 11 variables which have 2, 3, 3, 3, 6, 6, 6, 9, 6, 4, 9, 9, 12, 18, 12, 18, 18, 12, 12, 18, 27, 18, 36, 36, 36, 24, 36, 24, 54, 54, 72, 72, 72, 108, and 144 solutions, respectively.

Input	c_{SM} (Projective Degree)
Codimension 1 in with 2 gens. Dim. 3 X_{Σ_1}	0.3s
Codimension 1 in with 3 gens. Dim. 3 X_{Σ_1}	2.0s
Codim. 2 in $\mathbb{P}^2 \times \mathbb{P}^2$	0.3s
Codim. 2 in $\mathbb{P}^6 \times \mathbb{P}^2$ with degree $(3,0),(0,2)$ eqs.	3.9s
Codim. 2 in $\mathbb{P}^5 \times \mathbb{P}^3$ with degree $(2,1)$ and $(1,1)$ eqs.	12.4s
Codim. 2 in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3$ with degree $(2,1,0)$ and $(0,1,2)$ eqs.	4.8s
Codim. 3 in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3$ with degree $(2,1,0),(0,1,2),(1,2,0)$ eqs.	52.4s

Future/Current Work

- Try using the algorithm on examples arising from applications and see how it performs there.
- Investigate possible improvements to the M2 implementation of the algorithm.
- Extend the algorithms presented here to compute the c_{SM} class and Segre class of a subscheme of any smooth complete toric variety.
- Investigate whether one may consider the c_{SM} class as a more refined version of the Euler characteristic in a precise sense for subschemes of more general varieties (i.e. toric varieties).
- Code on github (`https://github.com/Martin-Helmer/char-class-calc`), will soon be added to the ”CharacteristicsClasses” M2 package.

Thank you for listening!

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