Topological Invariants and the Maximum Likelihood Degree of a Toric Variety

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Context

- N. Budur, B. Wang (i.e. "Bounding the maximum likelihood degree") relate the maximum likelihood (ML) degree to the degree of a (very affine) Guass map associated to a subvariety of the complex torus $(\mathbb{C}^*)^n$.
- Separately, the projective conormal variety has been studied in many settings and has deep connections to many invariants studied in intersection theory.
- These connections often lead to simple expressions for invariants (such as degrees of projection maps).
- For projective toric varieties many invariants of interest have simple combinatorial expressions; we will focus on this case.

Goal

- We wish to determine under what conditions the ML degree can be linked to the projective conormal variety.
- We see that when $X = X_A$ is a projective toric variety we can give a precise notion of an embedding of its affine cone $\tilde{X}_A$ in $\mathbb{C}^n$ via general coordinates that will let us link these objects.
- This allows us to use ideas from the intersection theory of projective algebraic varieties to study the ML degree.

The Guass map from subvarieites of $(\mathbb{C}^*)^n$

Let $\mathfrak{X}$ be a subvariety (or an analytic subset) of the complex torus $(\mathbb{C}^*)^n$. The tangent space to $\mathfrak{X}$ at a point $p \in \mathfrak{X}$ can be identified with a vector subspace $\Theta_{\mathfrak{X},p} \subset \mathbb{C}^n$.

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Let $\mathrm{Gr}(n, n-1)$ denote the Grassmann variety of $(n-1)$-dimensional vector subspaces in $\mathbb{C}^n$.

The Guass map is the projection $\gamma : \mathcal{P} \to \mathrm{Gr}(n, n-1)$ where

$$\mathcal{P} = \overline{\{(p, \alpha) \in \mathfrak{X}_{\mathrm{reg}} \times \mathrm{Gr}(n, n-1) \mid L_\alpha \supseteq \Theta_{\mathfrak{X},p}\}} \subseteq \mathfrak{X} \times \mathrm{Gr}(n, n-1)$$

and L_α is the vector subspace corresponding to $\alpha \in \mathrm{Gr}(n, n-1)$.

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N. Budur, B. Wang relate $\deg(\gamma)$ (that is $\#\gamma^{-1}(\alpha)$ for a general $\alpha \in \gamma(\mathcal{P}) \subset \mathrm{Gr}(n, n-1)$) to the ML degree.

Goal: relate $\deg(\gamma)$ to invariants of a projective variety (associated to $\mathfrak{X}$).

Projective Conormal varieties and Polar degrees

Let $X \subset \mathbb{P}^{n-1}$ be a projective variety, it's conormal variety is

$$\mathrm{Con}(X) = \overline{\{(p, L) \mid p \in X_{\mathrm{reg}} \text{ and } L \supseteq T_p X\}} \subset \mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^{\vee}.$$

Let $c = \mathrm{codim}(X)$, $\mathcal{J}$ be the $(c+1) \times (c+1)$-minors of the matrix $[J(X)\ y]^T$.

The ideal of $\mathrm{Con}(X)$ is $\mathcal{K} = (I_X + \mathcal{J}) : (I_{\mathrm{Sing}(X)})^{\infty}$.

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It's class in the Chow ring $A^*(\mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^{\vee}) \cong \mathbb{Z}[H, h]/(H^n, h^n)$ is

$$[\mathrm{Con}(X)] = \delta_0 H^{n-1} h + \cdots + \delta_{n-2} H h^{n-1}.$$

The integers $\delta_0 = \delta_0(X), \ldots, \delta_{n-2} = \delta_{n-2}(X)$ are the polar degrees of X.

In the language of books such as:

E. Miller, and B. Sturmfels: *Combinatorial commutative algebra*

the polar degrees are the multidegree of the bigraded ideal $\mathcal{K}$.

Polar degrees of toric varieties (and the Chern-Mather class)

Fix an integer $d \times n$-matrix $A = (a_1, a_2, \ldots, a_n)$ of rank d with $(1, 1, \ldots, 1)$ in its row space. Each column vector a_i represents a monomial $t^{a_i} = t_1^{a_{1i}} t_2^{a_{2i}} \cdots t_d^{a_{di}}$. The *affine toric variety* $\tilde{X}_A$ is the closure of $\{(t^{a_1}, \ldots, t^{a_n}) \in \mathbb{C}^n : t \in (\mathbb{C}^*)^d\}$.

Write $X_A \subset \mathbb{P}^{n-1}$ for the *projective toric variety* with the same parametrization.

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Theorem (M. Helmer - B. Sturmfels, 2016)

The polar degrees of the projective toric variety X_A are

$$\delta_i(X_A) = \sum_{j=i+1}^{d} (-1)^{d-j} \binom{j}{i+1} V_{j-1} \qquad \text{for } i = 0, 1, \ldots, d-1,$$

where V_j is the sum of the Chern-Mather volumes of all j-dimensional faces of $P = \mathrm{conv}(A)$.

If X_A is smooth these are the normalized lattice volumes.

Theorem

Let $\tilde{X}$ be an affine cone in $\mathbb{C}^n$ corresponding to a projective variety $X \subset \mathbb{P}^{n-1}$ and let $\mathfrak{X} = \tilde{X} \cap (\mathbb{C}^*)^n$ be the very affine part of $\tilde{X}$. Fix general coordinates (i.e. coordinate transformations chosen from a suitable Zariski open dense set) for the embedding of X in $\mathbb{P}^{n-1}$, and correspondingly for the embeddings of $\tilde{X}$ in $\mathbb{C}^n$ and $\mathfrak{X}$ in $(\mathbb{C}^*)^n$. Writing $\mathfrak{X}$ in these coordinates let

$$\gamma : \overline{\{(p,\alpha) \in \mathfrak{X}_{\mathrm{reg}} \times \mathrm{Gr}(n, n-1) \mid L_\alpha \supseteq \Theta_{\mathfrak{X},p}\}} \to \mathrm{Gr}(n, n-1)$$

be the Guass map. Then we have that

$$\deg(\gamma) = \delta_{n-1}(X).$$

General Coordinates for an embedded Toric Variety

Let $X_A \subset \mathbb{P}^{n-1}$ be the projective toric variety defined by an integer $d \times n$-matrix A.

The principal A-determinant variety $V(E_A) = \bigcup_{\substack{\alpha \text{ a face} \\ \text{of conv}(A)}} X^{\vee}_{A\cap\alpha}$.

Theorem

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$ be such that $\lambda \notin V(E_A)$ and let $\lambda \cdot \tilde{X}_A$ denote the change of coordinates on $\tilde{X}_A$ induced by the natural torus action $\lambda \cdot (x_1, \ldots, x_n) = (\lambda_1 x_1, \ldots, \lambda_n x_n)$; similarly let γ be the Guass map associated to $\mathfrak{X} = \lambda \cdot \tilde{X}_A \cap (\mathbb{C}^*)^n$ in $(\mathbb{C}^*)^n$. Then $\lambda \cdot \tilde{X}_A$ is in general coordinates and

$$\deg(\gamma) = \delta_{n-1}(X) = \mathrm{Vol}(P),$$

where $P = \mathrm{conv(A)}$.

Euler Characteristic

In "The Gauss map and a noncompact Riemann-Roch formula for constructible sheaves on semiabelian varieties" J. Franecki and M. Kapranov relate the (signed) topological Euler characteristic of $\mathfrak{X} \subset (\mathbb{C}^*)^n$ to $\deg(\gamma)$.

In general coordinates, these results, along with the previous result and the combinatorial description of the polar degrees of projective toric varieties gives the following.

Corollary

Let X_A be the projective toric variety defined by a $d \times n$ integer matrix A, let $\tilde{X}_A$ be the corresponding affine cone and let $P = \mathrm{conv}(A)$. Also let $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$ be such that $\lambda \notin V(E_A)$ and set $\mathfrak{X}_\lambda = (\lambda \cdot \tilde{X}_A) \cap (\mathbb{C}^*)^n$. We have that the (signed) topological Euler chacteristic of $\mathfrak{X}_\lambda$ is given by

$$\deg(\gamma) = (-1)^d \chi(\mathfrak{X}_\lambda) = \delta_{n-1}(X_A) = \mathrm{Vol}(P). \qquad (1)$$

Relation to ML Degree

A result of J. Huh ("The maximum likelihood degree of a very affine variety") relates the signed Euler characteristic of a smooth subset $\mathfrak{X}$ of $(\mathbb{C}^*)^n$ to the maximum likelihood degree.

Showing that $\mathfrak{X}_\lambda = (\lambda \cdot \tilde{X}_A) \cap (\mathbb{C}^*)^n$ is smooth, gives the following.

Corollary

Let X_A be the projective toric variety defined by an integer matrix A, let $\tilde{X}_A$ be the corresponding affine cone and let $P = \mathrm{conv}(A)$. Also let $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$ be such that $\lambda \notin V(E_A)$ and set $\mathfrak{X}_\lambda = (\lambda \cdot \tilde{X}_A) \cap (\mathbb{C}^*)^n$. We have that the maximum likelihood degree of $\lambda \cdot X_A$ is given by

$$(-1)^d \chi(\mathfrak{X}_\lambda) = \mathrm{MLdegree}(\lambda \cdot X_A) = \mathrm{Vol}(P). \tag{2}$$

Summary and Future Work

- We used classical projective invariants to study the degree of the very affine Guass map, and the ML degree.
- For toric varieties we gave explicit conditions for when these projective invariants give the degree of the very affine Guass map and the ML degree.
- The next question is to explore relations between the ML degree and non-toric varities in general coordinates.
- Thank you for listening!