$$B = \begin{bmatrix} b^{(1)} \\ \vdots \\ b^{(r)} \end{bmatrix}$$ row vectors of length n

$b = b^{(1)} = (b_1, \dots, b_n)$

$$\text{binomial}(b^{(1)}) = \prod_i x_i^{b_i^+} - \prod_i x_i^{b_i^-}$$

$$b^+ = \begin{cases} b_i & \text{if } b_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$b^- = \begin{cases} |b_i| & \text{if } b_i < 0 \\ 0 & \text{otherwise} \end{cases}$$

Thm (Division Alg). Work in $R = k[x_1, \dots, x_n]$.

Let $\{f_1, \dots, f_s\}$ be an ordered list of polynomials in R

Every $f \in R$ can be written as

$$f = \underbrace{q_1 f_1 + \dots + q_s f_s}_{\text{"quotient"}} + \underbrace{r}_{\text{remainder}}$$

$q_i, r \in R$.

s.t

$r = 0$

OR

r is a k-linear combo of monomials, none of which are divisible by $in_<(f_1), \dots, in_<(f_s)$

Further if $q_i f_i \neq 0 \Rightarrow in_<(f) \geq in_>(q_i f_i)$

Ideal membership

It is clear if $r=0$ above

$$f = q_1 f_1 + \cdots + q_s f_s \Rightarrow f \in I = \langle f_1, \ldots, f_s \rangle$$

But for an arbitrary generating set $r=0$ after division by $\{f_1, \ldots, f_s\}$ is only sufficient, not nessary for $f \in I$.

Ex) Let $f_1 = xy - 1$, $f_2 = y^2 - 1$ in $K_{lex}[x, y]$

Divide $f = xy^2 - x$ by $\{f_1, f_2\}$ gives

$$xy^2 - x = \underbrace{y}_{q_1}(xy-1) + \underbrace{0}_{q_2}(y^2-1) + \underbrace{(-x-1)}_{r}$$

Divide $\{f_2, f_1\}$

$$f = xy^2 - x = \underbrace{x}_{q_1}(y^2-1) + \underbrace{0}_{q_2}(xy-1) + \underbrace{0}_{r=0}$$

$\therefore f \in \langle f_1, f_2 \rangle$

Thm) Let $G = \{g_1, \ldots, g_t\}$ be a G.B. for $I \subseteq K[x_1, \ldots, x_n]$ (R) an ideal, $f \in R$. We have $f \in I$ if and only if the remainder on division of f by G is zero.

Def) (LCM) Let $f, g \in K_<[x_1, \ldots, x_n]$ be non-zero, $in_<(f) = x^a$, $in_<(g) = x^b$

then $x^\gamma = lcm(in_<(f), in_<(g))$, where $\gamma_i = \max(a_i, b_i)$, $i = 1, \ldots, n$

lcm for integer vectors

Def (S-polynomial) The S-polynomial of $f, g \neq 0$ in $k[x_1, \dots, x_n]$ is

$$S(f,g) = \operatorname{lcm}(\operatorname{in}_<(f), \operatorname{in}_<(g)) \cdot \left(\frac{f}{LT(f)} - \frac{g}{LT(g)} \right)$$

Set up to cancel lead terms

Notation Given a set $G = \{g_1, \dots, g_r\}$ an $f \in R$

write $f \% G$, or $\operatorname{rem}(f, G)$ for

the remainder of dividing f by G.

Thm/Alg (Buchberger's Algorithm) Let $I = \langle f_1, \dots, f_s \rangle \neq \{0\}$ in $k[x_1, \dots, x_n]$. Then a Gröbner basis for I can be computed the following alg.

Input: $F = \{f_1, \dots, f_s\}$

Output: a Gröbner basis $G = \{g_1, \dots, g_t\}$ for I with $F \subseteq G$

Procedure:

```
G = F
Repeat := true
while Repeat == true DO:
        G' := G
        For each pair {p,q}, p ≠ q, in G' DO:
            r := S(p,q) % G'
            If r ≠ 0 Then G = G ∪ {r}
     If G == G' Then ( Repeat = false : Return G )
```

ACC = Ascending Chain Condition, for ideals $I_j \subseteq R$

$I_1 \subseteq I_2 \subseteq \cdots I_n \subseteq \cdots$

this stabilizes, i.e. $I_j = I_{j+1}$ for all $j \geq N$.