Def | Let $\Sigma \subset N_{\mathbb{R}}$ be a rational fan.

The toric variety X_Σ associated to Σ is the collection $\{ V_\sigma \mid \sigma \in \Sigma \}$ of affine toric varieties for any $\tau = \sigma_1 \cap \sigma_2$ the glueing is given by the inclusion isomorphism, i.e. include the isomorphic copy of V_τ in V_{σ_1} and V_{σ_2}

whenever $\underset{\uparrow \text{a face}}{\tau} \subset \overset{\swarrow \text{a cone}}{\sigma}$ $\quad V_\tau \subseteq V_\sigma$

The minimal cone in Σ is 0 (Since Σ is pointed)

and $0^\perp = M_{\mathbb{R}}$, so, $S_0 = M$

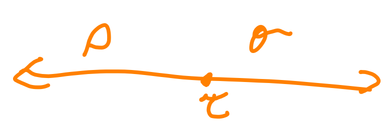
$$\therefore \; V_0 = \mathrm{Hom}_{sg}(M, \mathbb{C}) = \mathrm{Hom}_g(M, \mathbb{C}^*) \simeq (\mathbb{C}^*)^n$$

$$V_0 \subseteq V_\sigma \qquad \forall \sigma \in \Sigma$$

And hence each V_σ contains an isomorphic copy of the torus $V_0 \simeq (\mathbb{C}^*)^n$.

Example | $\Sigma = \{ \sigma, \rho, \tau \} \subseteq \mathbb{R}$ where $\quad X_\Sigma \cong \mathbb{P}^1$

$\sigma = \mathbb{R}_{\geq 0}$, $\rho = \mathbb{R}_{\leq 0}$, $\tau = \{0\}$



$\sigma^\vee = \sigma$, $\rho^\vee = \rho$, $\tau^\vee = \mathbb{R}$ $\qquad S_\tau = \mathbb{Z}$

$S_\sigma = \mathbb{N}$, $S_\rho = -\mathbb{N}$

$V_\sigma = \mathrm{Hom}_{sg}(\mathbb{N}, \mathbb{C}) \simeq \mathbb{C}$

$V_\rho = \mathrm{Hom}_{sg}(-\mathbb{N}, \mathbb{C}) \simeq \mathbb{C}$

for each $x \in \mathbb{C}$ $([x,1] \in \mathbb{P}^1)$ we have $f_x(n) = x^n$ is an s.g. hom

for each $y \in \mathbb{C}$ $([1,y] \in \mathbb{P})$ $g_y(n) = y^{-n}$

$V_\tau = \mathrm{Hom}_{sg}(\mathbb{Z}, \mathbb{C}) \simeq \mathbb{C}^*$

for each $z \in \mathbb{C}^*$ $h_z(r) = z^r$

So $X_\Sigma \simeq \mathbb{P}^1_{(x,y)} = \underbrace{(\mathbb{P}^1 - V(x))}_{\simeq V_\sigma} \cup \underbrace{(\mathbb{P}^1 - V(y))}_{\simeq V_\rho}$

and $\underbrace{V_\sigma \cap V_\rho}_{\simeq \mathbb{C}^* \simeq V_\tau}$ $\Leftrightarrow$ these meet at $[x, \frac{1}{x}]$, i.e $x \neq 0$

Projective toric Varieties from fans

Let $P \subseteq M_{\mathbb{R}}$ be a polytope with vertices in M

set $\underbrace{A}_{\text{finite set}} := P \cap M$

write $X_P := X_{A^+}$ ← the projective toric variety defined by monomial map given by A^+.

Let $\Sigma_P \subseteq N_{\mathbb{R}}$ be the outer normal fan of P

Recall: Each cone $\sigma \in \Sigma_P$ is the closure of the points in $N_{\mathbb{R}}$

that expose a given face F of P.

Each cone $\sigma \in \Sigma_P$ corresponds to a unique face

$$F = P_\sigma = P_w = \{x \in P \mid w \cdot x = h_P(w)\}$$

$\llcorner$ $w \in \text{rel.int}(\sigma) = \sigma -$ (all faces of σ)

Suppose $A = [a_0 \cdots a_m]$ is a $r \times (m+1)$ integer matrix with $\mathbb{Z}A = \mathbb{Z}^r$.

Goal | Define a map $V_\sigma \longrightarrow \mathbb{P}^m$ for any cone $\sigma \in \Sigma_P$

Choose any point $b \in P_\sigma \cap M$. By def P_σ

$$\langle w, b \rangle \geq \langle w, a_i \rangle \quad a_i \in \{a_0, \ldots, a_m\}$$

$\uparrow$ since w exposes P_σ $\therefore \langle w, b \rangle \geq w \cdot x \quad \forall x \in P$

$$\therefore \quad \langle w, b - a_i \rangle \geq 0 \quad \forall \; w \in \sigma$$

$$\therefore \quad b - a_i \in \sigma^\vee \cap M = S_\sigma$$

For any $f \in V_\sigma = \text{Hom}_{sg}(S_\sigma, \mathbb{C})$ define

$$\varphi_{b,\sigma}(f) := [f(b-a_0) : \cdots : f(b-a_m)] \in \mathbb{P}^m$$

Lemma | For any $b, b' \in P_\sigma \cap M$, $f \in V_\sigma$ we have $\varphi_{b,\sigma}(f) = \varphi_{b',\sigma}(f)$ in $\mathbb{P}^m$. Further for any $f \in V_\sigma$ $\varphi_{b,\sigma}(f) \in X_P$ and for any $x = [x_0 : \cdots : x_m] \in X_P$ with $x_i \neq 0$ $\forall i$ s.t. $a_i \in P_\sigma$ there exists a point $f \in V_\sigma$

with $X = \varphi_{b,\sigma}(f)$.

i.e $\{\varphi_\sigma(V_\sigma) \mid \sigma \in \Sigma\} \subseteq X_P$ and $X_P \subseteq \{\varphi_\sigma(V_\sigma) \mid \sigma \in \Sigma\}$

Proof | For any $b, b' \in P_\sigma \cap M$, $b - b' \in \sigma^\vee \therefore b - b' \in S_\sigma$

so for $\underset{\mathrm{Hom}_{sg}(S_\sigma, \mathbb{C})}{f \in V_\sigma}$, $f(b-b') \in \mathbb{C}$ $\left(\begin{array}{l} f(b-b') \neq 0 \\ \text{unless } f \equiv 0 \end{array}\right)$

For any $a_i \in \{a_0, \dots, a_m\} = A$ we have

$$f(b' - a_i) = f(b' - b + b - a_i) \underset{\text{Since } f \in \mathrm{Hom}_{sg}(S_\sigma, \mathbb{C})}{=} f(b' - b) \cdot f(b - a_i)$$

$\therefore$ as points in $\mathbb{C}^{m+1}$

$$\varphi_{b',\sigma}(f) = \overbrace{f(b' - b)}^{\in \mathbb{C}^*} \varphi_{b,\sigma}(f)$$

$$\therefore \quad \varphi_{b',\sigma}(f) = \varphi_{b,\sigma}(f) \text{ in } \mathbb{P}^m.$$

$\therefore$ Def of φ is independent of choice of b.

Now show $\varphi_{b,\sigma}(f) \in X_P = X_{A^+}$.

Recall that $X_{A^+} = V(I_{A^+})$ where I_{A^+} is the toric ideal

$$I_{A^+} = \langle z^u - z^v \mid A^+ u = A^+ v \rangle$$

So our goal is to show that $z^u - z^v$ vanishes at $\varphi_\sigma(f)$. Take $u, v \in \mathbb{N}^{m+1}$ s.t. $A^+ u = A^+ v$

Since the first row of A^+ is $(1, \dots, 1)$ we can split the condition into $Au = Av$ and $|u| = |v|$, $u = (u_0, \dots, u_m)$ $v = (v_0, \dots, v_m)$

$$(\varphi_{a_0}(f))^u = (f(b-a_0))^{u_0}\cdots(f(b-a_m))^{u_m} \quad \text{(By def)}$$
$$= f(u_0(b-a_0))\cdots f(u_m(b-a_m))$$
$$= f\left(\sum_{i=0}^m u_i(b-a_i)\right)$$
$$= f\left(\sum u_i b - \sum u_i a_i\right)$$
$$= f(|u|b - Au)$$
$$= f(|v|b - Av) = f\left(\sum v_i(b-a_i)\right) = (\varphi_{b,\sigma}(f))^v$$

$\therefore$ $z^u = z^v$ at $\varphi_\sigma(f)$ i.e. $z^u - z^v$ vanishes

$$\therefore \quad \varphi_\sigma(f) \in V(I_{A^+}) = X_{A^+}.$$

<u>Last Goal</u> Suppose $x = [x_0 : \cdots : x_n] \in X_{A^+}$ is s.t. $x_i \neq 0$ $\forall i$ s.t. $a_i \in P_\sigma \cap M$

Show $\exists$ a point $f \in V_\sigma$ with $x = \varphi_{b,\sigma}(f)$.

Choose some $b \in P_\sigma \cap M$, define a map

$b = a_j$

$$f : \mathbb{N}\{b-a_i \mid i = 0,\dots,m\} \to \mathbb{C} \quad \text{by}$$

$f(b-a_i) = \dfrac{x_i}{x_j}$ and extend by linearity

$x_j \neq 0$ since $b = a_j \in P_\sigma \cap M$

Also note that since $x \in X_{A^+}$ then $x^u = x^v$ whenever $A^+u = A^+v$, hence any time we have u, v with

$$\sum u_i(b-a_i) = \sum v_i(b-a_i)$$

then $f\left(\sum u_i (b_i - a_i)\right) = f\left(\sum v_i (b - a_i)\right)$

$\therefore$ f is well-defined

and is a s.g. hom from $G := \mathbb{N}\{b - a_i \mid i\}$ to $\mathbb{C}$

$G \subseteq S_\sigma$, so if $S_\sigma = \mathbb{N}\{b - a_i\}$ then we are done.

Since $\mathbb{C}$ is alg. closed we can extend the s.g. hom

$G \to \mathbb{C}$ to $S_\sigma \to \mathbb{C}$

Roughly if we take $\{\beta_1, \dots, \beta_\ell\}$ are generators for S_σ

$S_\sigma = \mathbb{N}\{\beta_1, \dots, \beta_\ell\}$

then $b - a_i = \sum n_j \beta_j$

then $\dfrac{x_i}{x_j} = f(b - a_i) = f\left(\sum n_j \beta_j\right) = \prod f(\beta_\ell)^{n_\ell}$

we can solve for the $f(\beta_1), \dots, f(\beta_\ell)$ since

$\mathbb{C}$ is alg. closed. $\blacksquare$

From the Lemma we have a well defined map

$$\varphi_\sigma : V_\sigma \longrightarrow X_P \subseteq \mathbb{P}^m$$

s.t. $\varphi_\sigma(V_\sigma) =$ set of points of X_P

with $x_i \neq 0$ if $a_i \in P_\sigma \cap M$

Lemma | Let $\sigma, \tau \in \Sigma$ be cones with $\tau \subseteq \sigma$

For any $f \in V_\tau$ we have $\varphi_\tau(f) = \varphi_\sigma(f)$

where $\varphi_\sigma(f)$ means $\varphi_\sigma(i(f))$

where $i : V_\tau \longrightarrow V_\sigma$

Proof: If $f \in V_\tau \Rightarrow f \in V_\sigma$ (via inclusion) is obtained by restricting f from S_τ to S_σ.

Taking $b \in P_\tau \cap M \subseteq P_\sigma \cap M$

we have $\{ b - a_i \mid i = 0, \dots, m \} \subseteq S_\sigma \subset S_\tau$

so $\varphi_{b,\tau}(f) = \varphi_{b,\sigma}(f)$ ∎

Let P a lattice polytope, Σ_P its normal fan

we have a map

$$\varphi_P : X_\Sigma \longrightarrow X_P = X_{A^*}$$

given by the collection of maps

$$\{ \varphi_\sigma \mid \sigma \in \Sigma \}$$

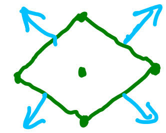
[Each maps from the corresponding part of $\{ V_\sigma \mid \sigma \in \Sigma \}$]

which agrees on the inclusions $V_\tau \subseteq V_\sigma$ given by inclusion of faces $\tau \subseteq \sigma$.

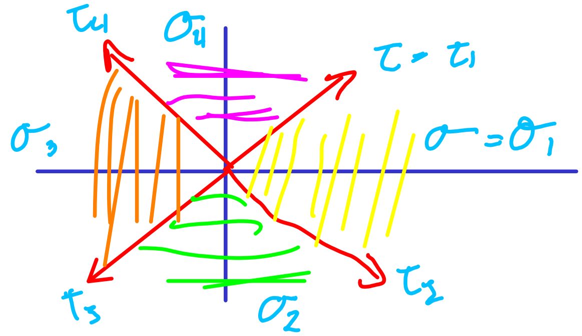
This map is in fact an isomorphism of algebraic varieties

so $X_\Sigma \simeq X_P = X_{A^*}$.

Consider $P = \mathrm{conv}(A)$, $A = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}$ =



Its normal fan is



$\tau_i = \mathbb{R}_{\geq}\begin{pmatrix} \pm 1 \\ \pm 1 \end{pmatrix}$

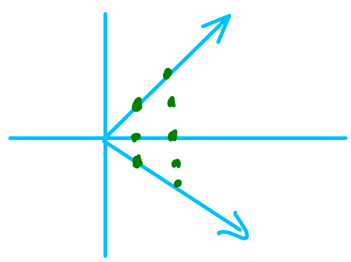
$\tau = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\sigma_i = \tau_i + \tau_{i+1}$ $\qquad \sigma_i \cong \sigma$, $\tau_i \cong \tau$

$\sigma = \mathbb{R}_{\geq 0}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathbb{R}_{\geq 0}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \tau + \tau_2$

$\Sigma = \Sigma_P = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \tau_1, \tau_2, \tau_3, \tau_4, 0\}$

V_σ $\qquad \sigma = \sigma^\vee$ =



$S_\sigma = \sigma^\vee \cap \mathbb{Z}^2$

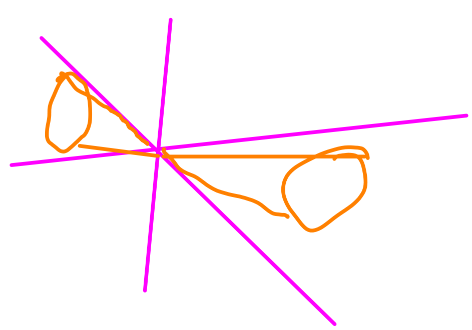
$S_\sigma = \mathbb{N}A$

where $A = \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \\ \begin{matrix} z_1 & z_2 & z_3 \end{matrix} \end{array}$

$\therefore$ $V_\sigma \cong X_A$ = closure in $\mathbb{C}^3$ of image

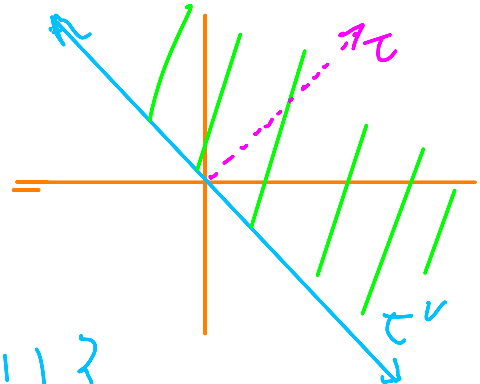
$\varphi_A : (s,t) \longmapsto \left(\frac{s}{t}, st, s\right)$

$X_A = V(z_1 z_2 - z_3^2)$.=



$\boxed{V_\tau}$ $\qquad \tau = \mathbb{R}_{\geq 0}\begin{pmatrix}1\\1\end{pmatrix}$

$$\tau^\vee = \{(u,v) \in \mathbb{R}^2 \mid u+v \geq 0\} =$$



$$\tau^\vee \cap \mathbb{Z}^2 = \mathbb{N}\left\{\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\}$$
$$= \mathbb{N}B$$

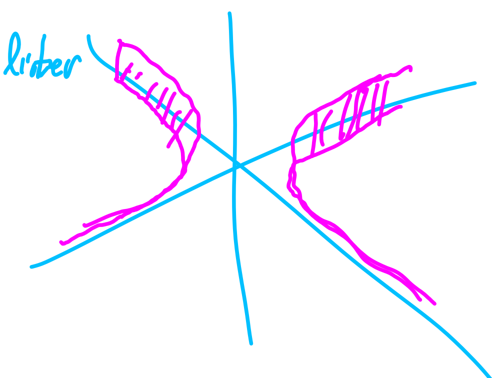
$$B = \begin{array}{c} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \\ \begin{matrix} z_1 & \tilde{z}_2 & z_3 \end{matrix} \end{array}$$

$$V_\tau \simeq X_B$$

$$\varphi_B : (s,t) \longmapsto (st^{-1}, s^{-1}t, s)$$

$$X_B = V(z_1\tilde{z}_2 - 1)$$

L= hyperbolic cylinder



Now the glueing:

$$\begin{matrix} V_0 \subseteq V_\tau \subseteq V_\sigma \\ \wr\!\| \\ (\mathbb{C}^*)^2 \end{matrix}$$

$(\mathbb{C}^*)^2 \subseteq V_\sigma$

$V_\sigma \setminus V(z_1 z_2 z_2)$

The boundry in V_σ is $V_\sigma \cap V(z_1z_2z_3)$

$= \underbrace{V(z_1,z_3)}_{z_2 \text{ axis}} \cup \underbrace{V(z_2,z_3)}_{z_1\text{-axis}}$

$(\mathbb{C}^*)^2 \subseteq V_\tau$

$= V_\tau \setminus V(z_1\tilde{z}_2 z_3)$

The Boundry of $(\mathbb{C}^*)^2$ in V_τ

is $V_\tau \cap V(z_1\tilde{z}_2 z_3) = V(z_3, z_1\tilde{z}_2 - 1)$

$t \neq 0$ on this hyperbola

$z_1 = t$, $\tilde{z}_2 = \frac{1}{t}$

on V_σ

$\frac{z_2}{z_1} = t^2$

so $t = 0 \longleftrightarrow z_1$-axis

so V_t (when $t \neq 0$) $\longleftrightarrow V_\sigma - (z_1 \text{ axis})$

and the boundary of $(\mathbb{C}^*)^2$ in V_t is identified with the z_2-axis in V_σ