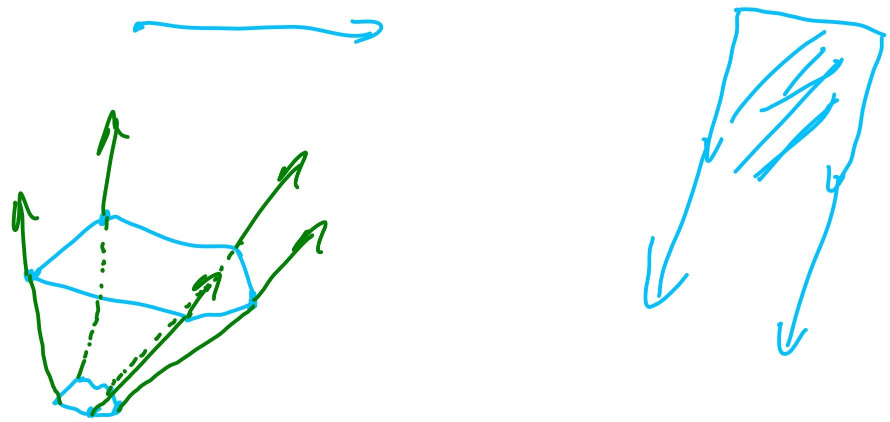
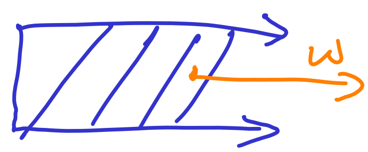
Def| (Polyhedron): A polyhedron is the intersection of finitely many half-spaces. A polytope is a bounded polyhedron

Ex| (some unbounded polyhedra):



Polyhedra also have a support function, $h_P(w)$, but the values are in $\mathbb{R} \cup \{\infty\}$ when P is unbounded in the direction of w

$$h_P(w) = \infty$$



$$h_P(w) = \infty = \max(w \cdot x \mid x \in P)$$

with this h_P a polyhedron is

$$P = \bigcap_{w \in \mathbb{R}^n} \{ x \in \mathbb{R}^n \mid w \cdot x \le h_P(w) \}$$

Now we can define a convex cone as a polyhedron where each supporting hyperplane contains the origin

Alt. a cone σ is a polyhedron whose support func

$h_\sigma(w) =$ Either 0 or ∞

$$\sigma = \bigcap_{w \in \mathbb{R}^n} \{x \in \mathbb{R}^n \mid w \cdot x \leq 0\}$$

Note that a Half-space $H_w := \{x \in \mathbb{R}^n \mid w \cdot x \leq 0\}$ forms a semi-group, i.e. $y, z \in H_w$

then $w \cdot y \leq 0$, $w \cdot z \leq 0 \Rightarrow w \cdot y + w \cdot z = w \cdot (y+z) \leq 0$

$y+z \in H_w$

Element is in the bounding hyperplane

$$w \cdot x = 0$$

If $w \cdot y = 0 \Rightarrow -w \cdot y = 0$ and $w(y-y) = 0$

$\qquad\qquad\quad \| $

$\qquad\quad w(-y) = 0 \qquad \therefore -y \in H_w$

Since a cone is

$$\sigma = \bigcap_{w \in \mathbb{R}^n} H_w$$

$\therefore \sigma$ is a semi-group, under addition

linearity space of $\sigma = \bigcap_{w \in \mathbb{R}^n} \underbrace{\{x \in \mathbb{R}^n \mid w \cdot x = 0\}}_{\text{Boundry hyperplane}}$

$=$ Set of invertible elements of σ.

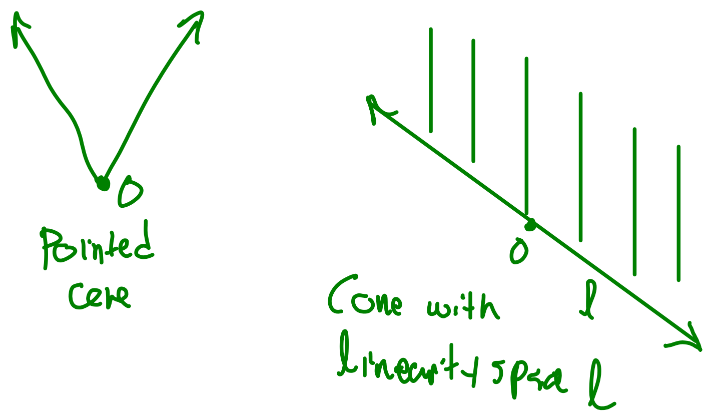
If lin. space of $\sigma = 0 \in \mathbb{R}^n$

we say σ is a pointed cone strictly/strongly convex.

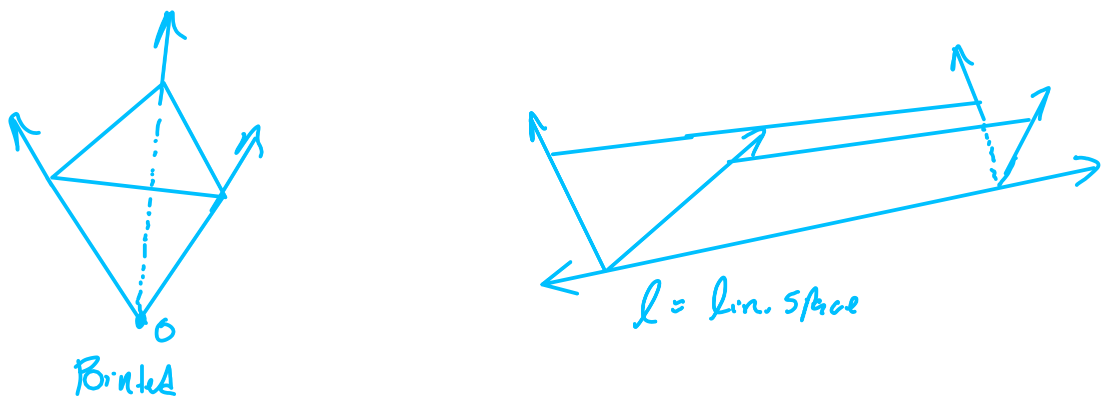
The linearity space of σ is its minimal face.

Ex)

$\mathbb{R}^2$



$\mathbb{R}^3$



= the face of σ contained in all faces of σ.

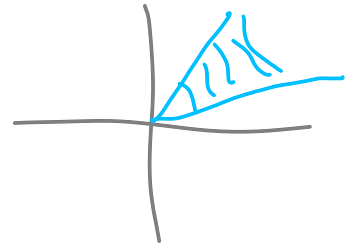
The minimal face of a pointed cone is 0

For a pointed cone σ its rays are its one dimensional faces

$$\rho = \mathbb{R}_{\geq 0}\, x \quad \text{for any } x \neq 0 ,\ x \in \rho$$

$\sigma =$ Sum of its rays $\rho \in \sigma$

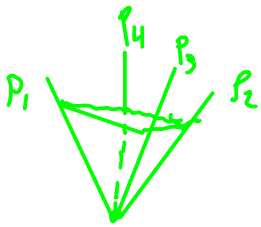
i.e if σ has rays $\{\rho_1, \dots, \rho_n\}$



$$\sigma = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_n$$

where $v_i \in \rho_i$

Eg. $\sigma = \underbrace{\mathbb{R}_{\geq 0}\begin{pmatrix}2\\1\\0\end{pmatrix}}_{\rho_1} + \underbrace{\mathbb{R}_{\geq 0}\begin{pmatrix}2\\-1\\0\end{pmatrix}}_{\rho_2} + \underbrace{\mathbb{R}_{\geq 0}\begin{pmatrix}2\\0\\1\end{pmatrix}}_{\rho_3} + \underbrace{\mathbb{R}_{\geq 0}\begin{pmatrix}2\\0\\-1\end{pmatrix}}_{\rho_4}$



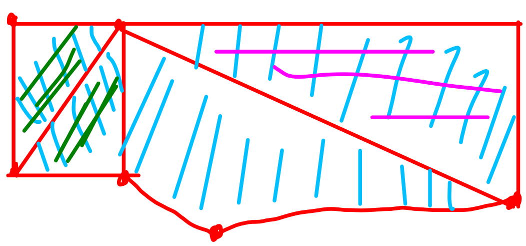
Eg. $\{x \mid w \cdot x \leq 0\}$

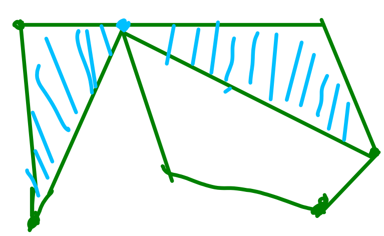
$w \cdot x = 0$

$= \underbrace{\mathbb{R}_{\geq 0}\begin{pmatrix}2\\1\\0\end{pmatrix}}_{\rho_1} + \underbrace{\mathbb{R}\begin{pmatrix}2\\-1\\0\end{pmatrix}}_{\rho_2} + \underbrace{\mathbb{R}_{\geq 0}\begin{pmatrix}0\\0\\1\end{pmatrix}}_{\rho_3} + \underbrace{\mathbb{R}_{\geq 0}\begin{pmatrix}0\\0\\-1\end{pmatrix}}_{\rho_4}$

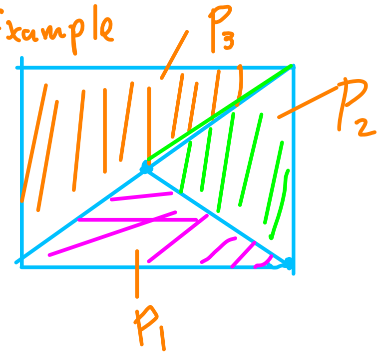
Def (Polyhedral complex). A <u>polyhedral complex</u> $\mathcal{P}$ is a collection of polyhedra in $\mathbb{R}^d$ s.t. every face of every polyhedron $P \in \mathcal{P}$ is another polyhedron and for $P, P' \in \mathcal{P}$, $P \cap P'$ is a face of P and P'.

Ex.





Non-Example



$P_3 \cap P_2$ is not a face of P_3

Ex) A Polytope, together with all its faces is a polyhedral complex.

Ex) Take a polytope P and a point $o \in P$.

For every face F of P s.t. $o \notin F$

Set Pyramid(F, o) = Pyramid with base F and apex o

Then the collection of:

- These Pyramids
- their bases
- o

is a polyhedral complex which subdivides P

Def) The support of a polyhedral complex $\mathcal{P}$ is the union of all $P \in \mathcal{P}$

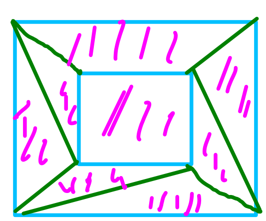
When $\text{Supp}(\mathcal{P}) = P$ = a polyhedron we say that $\mathcal{P}$ is a subdivision of P.

when $\mathrm{supp}(\mathcal{P}) = P$ = a polytope

$$\mathrm{Vol}_n(P) = \sum_{Q \in \mathcal{P}} \mathrm{Vol}_n(Q)$$

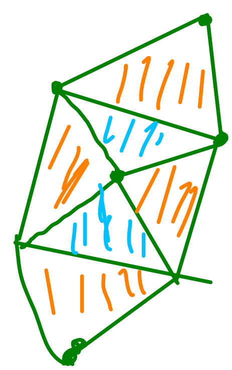
when every polytope in a polyhedral complex $\mathcal{P}$ is a simplex we say $\mathcal{P}$ is a triangulation of $\mathrm{supp}(\mathcal{P}) = P$.

Ex.



polyhedral subdivision not triangulation

triangulation

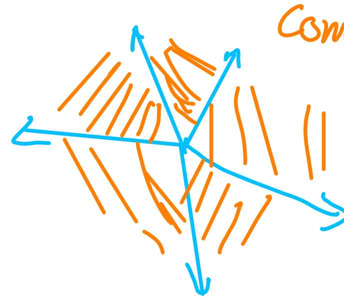


Def (Fan). A polyhedral complex Σ in $\mathbb{R}^d$ is called a fan if all polyhedra $C \in \Sigma$ are cones. If $\mathrm{Supp}(\Sigma) = \mathbb{R}^d$, Σ is called complete.

Ex

A fan, not complete

Complete fan in $\mathbb{R}^2$



Given a polytope $P \subseteq \mathbb{R}^n$ define an equivalence relation on $(\mathbb{R}^n)^V \cong \mathbb{R}^n$ by $v \sim w$

iff $P_v = P_w$

$\uparrow$ = Face of P exposed by $\underline{w}$

$= \{x \in P \mid w \cdot x = h_P(w)\}$

Face exposed by a vector $w \in \mathbb{R}^n$

$P_w = \{x \in P \,/\, w \cdot x = h_P(w)\}$

These eq. classes give rise to the Normal cone to the face $F = P_w$ i.e. $N_P(F) =$ closure of the set of points in the eq. class of P_w

The <u>Normal cone</u> of a face $F = P_w$

$$N_P(F) = \{ u \in \mathbb{R}^n \mid u \cdot (x - f) \leq 0 \quad \forall x \in P \text{ and some } f \in \operatorname{relint}(F) \}$$

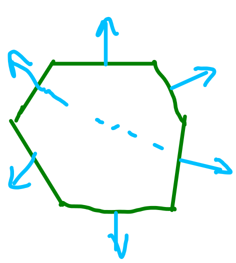
Relative interior of a polytope

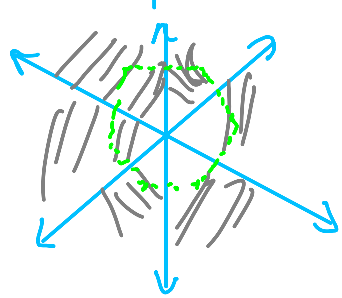
$$\operatorname{relint}(F) = \{ x \in F \mid \forall y \in F \ \exists \lambda > 1 \text{ s.t. } \lambda x + (1-\lambda) y \in F \}$$

For a polytope P the collection $N_P(\tau)$ for τ a face of P forms the (outer) normal fan $\mathcal{F}$ of P. This is a complete fan.

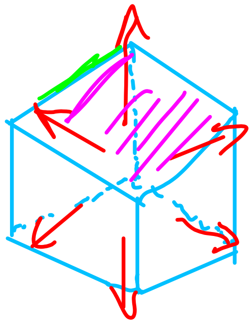
The rays of this fan expose the facets of P.

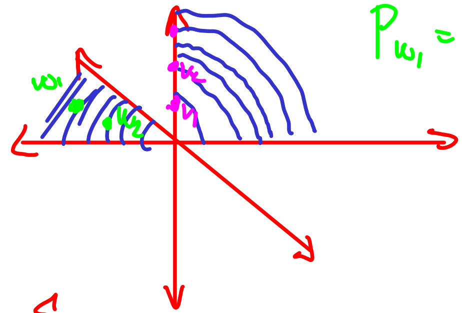
Ex)





Ex)





$P_{w_1} = P_{w_2} =$ edge of P

$P_{v_1} = P_{v_2} =$ Facet of P

$\Sigma =$ Normal fan of cube

The rays of Σ are $\begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \pm 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix}$

Each exposes a facet

The two dim cones in Σ ar the positive spans of any pair of rays, these expose an edge of P

The orthant are pos. span of 3 rays and expose a vertex of P.

Toric Varieties From Fans

Goal: Construct a toric variety by glueing affine toric varieties along common subsets

Work with torus $(\mathbb{C}^*)^n$

$$N = \mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n) \simeq \mathbb{Z}^n = \text{group of cocharacters}$$

$$M = \mathrm{Hom}((\mathbb{C}^*)^n, \mathbb{C}^*) \simeq \mathbb{Z}^n = \text{group of characters}$$

$$N_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} N \simeq \mathbb{R}^n = \text{real v. space spanned by cocharacters}$$

$$M_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} M \simeq \mathbb{R}^n = \mathbb{R}\text{ v. space spand by characters}$$

Write $\langle \cdot , \cdot \rangle : N_{\mathbb{R}} \times M_{\mathbb{R}} \to \mathbb{R}$

$$n \times m \mapsto n \cdot m$$

A (rational) fan $\Sigma \subseteq N_{\mathbb{R}}$ is a fan in $N_{\mathbb{R}}$ in which every cone is defined by inequalities coming from elements $a \in M$, i.e. the half spaces defining cones in Σ all have form

$$\{ w \in N_{\mathbb{R}} \mid \langle w, a \rangle \geq 0 \} \text{ for some } a \in M.$$

For a cone $\sigma \in \Sigma$ the linear span of σ is a rational linear space, i.e.

$$\mathrm{Span}_{\mathbb{R}}(\sigma) = \mathrm{Span}_{\mathbb{R}}(\sigma \cap N)$$

<u>Assumption</u> | we will assume all cones $\sigma \in \Sigma$ are pointed

This is not such a restriction:

In gen. all cones in a rational fan have some linearity space $L_{\mathbb{R}}$, this has a full rank sublattice

$$L = N \cap L_{\mathbb{R}}$$

Replace N by N/L, M by $L^{\perp}$ and every $\sigma \in \Sigma$ by its image in $N_{\mathbb{R}}/L_{\mathbb{R}}$

$\Rightarrow$ give a pointed fan.

Def (Dual Cone) Given a rational cone $\sigma \subset N_{\mathbb{R}}$ its dual cone is

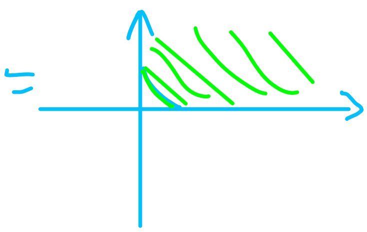
$$\sigma^\vee := \{ x \in M_{\mathbb{R}} \mid \langle w, x \rangle \geq 0 \ \forall w \in \sigma \}$$

The linearity space of $\sigma^\vee = \sigma^\perp$

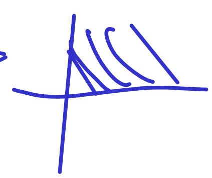
$$\dim(\sigma^\perp) = n - \dim(\sigma)$$

Ex

$$\sigma = \mathbb{R}_{\geq 0}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbb{R}_{\geq 0}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

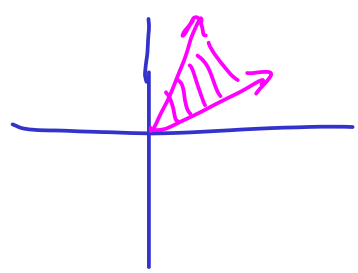
$=$ 

$$\sigma^\vee = \{ x \in \mathbb{R}^2 \mid x \cdot w \geq 0 \ \forall w \in \sigma \}$$

$= \sigma =$ 

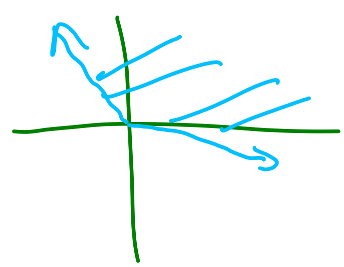
Ex

$$\sigma = \mathbb{R}_{\geq 0}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \mathbb{R}_{\geq 0}\begin{pmatrix} 2 \\ 1 \end{pmatrix} =$$

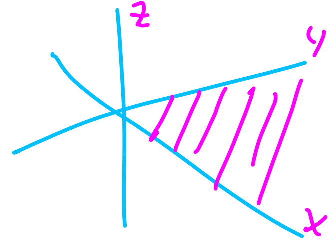


$$\sigma^\vee = \{ x \in \mathbb{R}^2 \mid x \cdot w \geq 0 \ \forall w \in \sigma \}$$

$$= \mathbb{R}_{\geq 0}\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \mathbb{R}_{\geq 0}\begin{pmatrix} -1 \\ 2 \end{pmatrix} =$$



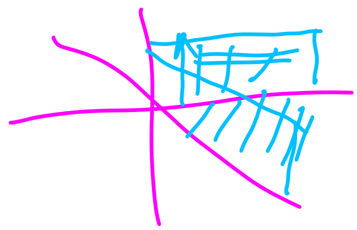
[Ex] $\sigma = \mathbb{R}_{\geq 0}\begin{pmatrix}1\\0\\0\end{pmatrix} + \mathbb{R}_{\geq 0}\begin{pmatrix}0\\1\\0\end{pmatrix} =$



$\sigma^\vee = \{ x \in \mathbb{R}^3 \mid w \cdot x \geq 0 \ \forall w \in \sigma \}$

$= \mathbb{R}_{\geq 0}\begin{pmatrix}1\\0\\0\end{pmatrix} + \mathbb{R}_{\geq 0}\begin{pmatrix}0\\1\\0\end{pmatrix} + \mathbb{R}_{\geq 0}\begin{pmatrix}0\\0\\1\end{pmatrix}$

$=$



Given a pointed rational cone $\sigma \subseteq N_{\mathbb{R}}$ set

$$S_\sigma := \sigma^\vee \cap M = \{ a \in M \mid \langle w, a \rangle \geq 0 \ \forall w \in \sigma \}$$

$\uparrow$ sub semigroup of M

The group of invertible elements of S_σ (i.e. the part of the linearity space of $\sigma^\vee$ in M)

is the free abelian group $\sigma^\perp \cap M$

$$\operatorname{rank}(\sigma^\perp \cap M) = n - \dim \sigma$$

Set

$$V_\sigma := \operatorname{Hom}_{sg}(S_\sigma, \mathbb{C}) = \text{set of semi-group homomorphisms from } S_\sigma \text{ to } \mathbb{C}.$$

Similar to our first construction of toric varieties

(Plus a thm of Hilbert which tells us S_σ is finitely generated as a semi-group)

$S_\sigma \cong \mathbb{N}A$ for some gen. set $A = \{a_1, \dots, a_m\}$ of S_σ.

$$\mathrm{Hom}_{sg}(S_\sigma, \mathbb{C}) \cong \mathrm{Spec}(\mathbb{C}[S_\sigma])$$

This is exactly the affine toric variety from beginning just we don't know 'a priori', what A is s.t. $S_\sigma = \mathbb{N}A$.

i.e. we don't know what $\mathbb{C}^m$ this embeds in

To embed in $\mathbb{C}^m$ we just find gen. set of S_σ.