Recall: Last time we showed that

$$h_{I_{A^+}}(d) \leq E_{conv(A)}(d) \quad \forall d$$

Current goal: Find a lower bound for $h_{I_{A^+}}(d)$ $\forall d$ using semi-group inclusion

$S_A = \mathbb{R}_{\geq 0} A^+ \cap \mathbb{Z}^{r+1}$ = semi-group of all integer points in pos. span of A^+.

To get our inclusion seek to construct last time

$$v_A \in \mathbb{N}A^+ \text{ s.t. } v_A + S_A \subseteq \underbrace{\mathbb{N}A^+ \subseteq S_A}_{\text{last time}}$$

$B \subseteq S_A$ = set of points $b \in \mathbb{Z}^{r+1}$ s.t.

$$b = \sum_{i=1}^{m} \lambda_i (1, a_i) \quad , \lambda_i \in \mathbb{Q} \quad , 0 \leq \lambda_i < 1$$

For each $b \in B$ re-express b in the form $\mathbb{Z}A^+ = \mathbb{Z}^{r+1}$

$$b = \sum_{i=1}^{m} \beta_i(b) \underbrace{(1, a_i)}_{\in \mathbb{Z}^{r+1}} \quad , \underbrace{\beta_i(b) \in \mathbb{Z}}_{\text{Exists since } b \in S_A \text{ which means } b \in \mathbb{Z}^{r+1}}$$

(not unique) Fix such a choice

Let $-v$, with $v \geq 0$ be an integer lower bound for all coefficients $\beta_i(b)$ appearing for any $b \in B$ (finite)

Ex) $A = [0\ 2\ 3]$, $B = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$, $A^+ = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B_1\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)} = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \in A^+$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$V = 1$$

Define $\quad v_A := v \sum_{i=1}^{m} (1, a_i)$

Eg) $A = [0, 2, 3]$ $\quad v_A = \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 1 \cdot \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right)$

Lemma) $v_A + S_A \subseteq \underbrace{\mathbb{N}A^+ \subseteq S_A}_{\text{last time}}$ $\quad S_A = \mathbb{R}_{\geq 0} A^+ \cap \mathbb{Z}^{n+1}$

Proof) show $v_A + S_A \subseteq \mathbb{N}A^+$.

$u \in v_A + S_A$.

$\Rightarrow \quad u - v_A \in S_A \quad \therefore$

$$u - v_A = \sum_{i=1}^{m} \alpha_i (1, a_i) \qquad \alpha_i \in \mathbb{Q}_{\geq 0}$$

For each $\quad \alpha_i = \underbrace{\lambda_i}_{\lambda_i \in [0,1) \cap \mathbb{Q}} + \underbrace{\gamma_i}_{\gamma_i \in \mathbb{N}}$

Then

$$u - v_A = \underbrace{\sum \lambda_i (1, a_i)}_{:= b \in B} + \underbrace{\sum \gamma_i (1, a_i)}_{:= c} \in \mathbb{N}A^+$$

since $\lambda_i \in [0,1) \cap \mathbb{Q}$

Since $b \in B$ we may use our fixed expression for b

$$b = \sum_{i=1}^{m} \beta_i(b)(1, a_i)$$

So then

$$w = v_A + b + c = v_A + \sum_{i=1}^{m} \beta_i(b)(1,a_i) + c$$

$$= \sum_{i=1}^{m} (\underbrace{\beta_i(b)}_{\in \mathbb{Z}} + \overset{\in \mathbb{N}}{\gamma})(1,a_i) + \underbrace{c}_{\in \mathbb{N}A^+}$$

Since $-\gamma \leq \beta_i(b) \Rightarrow \beta_i(b) + \gamma \geq 0 \quad \forall i$

$$\overbrace{\sum_{i=1}^{m} (\beta_i(b) + \gamma)(1,a_i)}^{\in \mathbb{N}A^+}$$

$\therefore \quad w \in \mathbb{N}A^+$ and

$$w = v_A + (\overbrace{b + c}^{u - v_A}) = v_A + u - v_A = u$$

$$\therefore \quad u \in \mathbb{N}A^+$$

$$\therefore \quad v_A + S_A \subseteq \mathbb{N}A^+$$

$\therefore \quad v_A + S_A \subseteq \mathbb{N}A^+ \subseteq S_A$

Recall that we had shown that the hilbert function was

$$h_{I_A^+}(d) = |dA| \quad \text{where}$$

dA = set of d-fold sums of vectors in A

This inclusion, restricting to points with d as the first coordinate gave

$$dA \subseteq d\,\mathrm{conv}(A) \cap \mathbb{Z}^n$$

$E_P : d \longmapsto |dP \cap \mathbb{Z}^r|$

Erhart Function of a polytope P

$\therefore$ with $P = \mathrm{conv}(A)$

$$h_{I_{A^+}}(d) \leq E_{\mathrm{conv}(A)}(d) \quad \forall d$$

$$\underbrace{\overbrace{V_A + S_A} \subseteq \mathbb{N}A^+} \subseteq S_A$$

Note that the first coordinate of V_A is $v|A|$

$\therefore$ vectors with 1^{st} coordinate d in $V_A + S_A$

are $(d - v|A|)$-fold sums of vectors in $V_A + S_A$

$\therefore$ we have

$$(d - v|A|)\,\mathrm{conv}(A) \cap \mathbb{Z}^r \subseteq dA$$

$A = [0, 2, 3]$, $\quad d = 4$, $\quad v = 1$

$|d - 1 \cdot 3| = 1 \qquad \mathrm{conv}(A) \cap \mathbb{Z}^r \subseteq 4A$

$0, 1, 2, 3 \qquad 4A = \{0, 1, 2, 3, \dots, 12\}$

$1 = 3 - 2$

$P := \mathrm{conv}(A)$

Putting this together

$$E_{\mathrm{conv}(A)}(d - v|A|) \leq h_{I_{A^+}}(d) \leq E_{\mathrm{conv}(A)}(d) \quad \forall d$$

Poly in d with $LC = \mathrm{vol}(P)$, $\deg = \dim(P)$

we know for a polytope P

$$E_P(d) = \text{Polynomial in } d$$

with

$$\deg(E_P) = \dim(P), \quad LC(E_P) = \mathrm{vol}(P)$$

Since this holds for all d, and $HP_{I_{A^+}}(d) = h_{I_{A^+}}(d)$ for large d

$\Rightarrow$ The hilbert poly. has the same degree and LC as the Erhart polynomial $E_P(d)$

$\therefore$ $\deg(HP_{I_{A^+}}(d)) = \dim(P) = r$

$LC(HP_{I_{A^+}}(d)) = Vol(conv(A))$

$\Downarrow$

$\deg(X_{A^+}) := r! \cdot LC(HP_{I_{A^+}}(d))$

Normalized volume

$\therefore$ $\deg(X_{A^+}) = r!\, vol(conv(A))$

$r = \dim(conv(A))$

Assuming $\mathbb{Z}A^+ = \mathbb{Z}^{r+1}$

Thus the generic number of solutions, $d(A)$, to the system

$$(*) \quad f_1(X) = \dots = f_r(x) = 0 \quad , \quad f_i(x) = \sum_{a_i \in \mathbb{Z}^r} c_{ij} x^{a_j}$$

$$d(A) = \deg(X_{A^+}) = \underbrace{r!\, vol(conv(A))}_{\text{Normalized volume}} \cdot [\mathbb{Z}A^+ : \mathbb{Z}^{r+1}]$$

Abstract Toric Varieties

Thus for $A = [a_1, \dots, a_m]$ and constructed embedded toric varieties $X_A \subseteq \mathbb{C}^m$, $X_{A^+} \subseteq \mathbb{P}^{m-1}$

Now we want to construct abstract toric varieties without reference to an embedding

← This both more general and more specific

since we obtain normal toric varieties in this manner

since no fixed ambient space

---

we had seen before how an affine toric variety $X_A \subseteq \mathbb{C}^n$ corresponds to a cone over $P = \mathrm{conv}(P)$

- we will obtain abstract toric varieties by glueing affine toric varieties of a cone, the glueing is defined by a fan.

Example $\mathbb{P}^1$ = Projective line

As a monomial map it's defined by

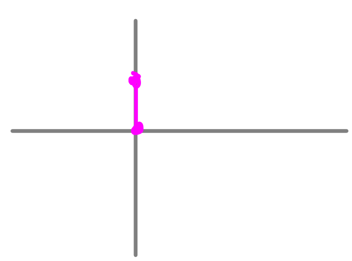
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \left(\text{or } \hat{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$\varphi_A : (t_1, t_2) \longmapsto [t_1 ; t_1 t_2]$$

$$\varphi_{\hat{A}} : (t_1, t_2) \longmapsto [t_1 ; t_2]$$

$$\mathbb{P}^1 = \overline{\varphi_A((\mathbb{C}^*)^2)} = \overline{\varphi_{\hat{A}}((\mathbb{C}^*)^2)}$$

$\mathrm{Conv}(A) = [0, 1]$ (on y-axis)



Write $[x:y]$ hom. coords on $\mathbb{P}^1$

The std. affine patches

$$\mathbb{C}_0 := \{[x:1] \mid x \in \mathbb{C}\} \subseteq \mathbb{P}^1$$

$$\mathbb{C}_\infty := \{[1:y] \mid y \in \mathbb{C}\} \subseteq \mathbb{P}^1$$

$$\mathbb{P}^1 = \mathbb{C}_0 \cup \mathbb{C}_\infty$$

$\mathbb{C}_0 \cap \mathbb{C}_\infty$ = Points of either patch where neither coord $= 0$

$$= \{[x:1] \mid x \in \mathbb{C}^*\}$$

$$= \{[1:\tfrac{1}{x}] \mid x \in \mathbb{C}^*\} \simeq \mathbb{C}^*$$

Alg. Organize this using subalgebras of $\mathbb{C}[x, x^{-1}]$

$$\mathbb{C}_0 \cong \overset{\mathbb{C}}{\cancel{\mathrm{Spec}(\mathbb{C}[x])}} \overset{\subseteq \mathbb{C}^2}{\longleftrightarrow} \mathrm{Hom}_{sg}(\mathbb{N}, \mathbb{C})$$

$$[x:1] \longmapsto f_x : \mathbb{N} \to \mathbb{C}$$

$$f_x : 0 \mapsto 1,\quad 1 \mapsto x$$

$$\mathbb{C}_\infty \simeq \mathrm{Hom}_{sg}(-\mathbb{N}, \mathbb{C})$$

$$[1,y] \longmapsto g_y : -\mathbb{N} \to \mathbb{C}$$

$$g_y : 0 \mapsto 1,\quad -1 \mapsto y$$

$$\mathbb{C}^* = \mathrm{Spec}(\mathbb{C}[x, x^{-1}]) \simeq \mathrm{Hom}_{sg}(\mathbb{Z}, \mathbb{C})$$

$$z \longmapsto h_z : 0 \mapsto 1,\quad 1 \mapsto z$$

$$h_z|_{\mathbb{N}} = f_x, \quad h_z|_{-\mathbb{N}} = g_y$$

Polytopes Again

Let $A = [a_1, \ldots, a_m]$

$$P := \mathrm{Conv}(A) = \left\{ \sum_{i=1}^{m} \lambda_i a_i \;\middle|\; \sum \lambda_i = 1,\ \lambda_i \geq 0 \right\}$$

closed and bounded

$\therefore$ given $w \in \mathbb{R}^m$ the lin. function $x \mapsto w \cdot x$ is

bounded on P and has a maximum on P

Let $h_P(w) :=$ max of lin func. $x \mapsto w \cdot x$ on P.

$$P_w := \{ x \in P \mid w \cdot x = h_P(w) \} \subseteq P$$

Face of P exposed by w,

$P_w = \mathrm{Conv}(A_w)$

where $A_w := \{ a \in \{a_1, \ldots, a_m\} \mid w \cdot a = h_P(w) \}$

Ex (Pyramid) Suppose P is a polytope with $\dim(P) = n-1$

and suppose $P \subseteq H_w$, $H_w =$ hyperplane in $\mathbb{R}^n$, $w \neq 0$

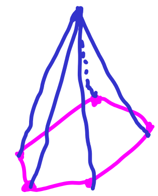
$$H_w = \{ x \in \mathbb{R}^n \mid w \cdot x = b \}$$

For any point $o \in \mathbb{R}^n - H$, the pyramid of o and P

is the convex hull of P and o

Q = Pyramid with base P apex 0

$$\mathrm{Vol}(Q) = \frac{1}{n}\, \overset{\text{height = dist from 0 to } H_w}{h}\, \mathrm{Vol}_{n-1}(P)$$



Half-spaces and supporting hyper plane

since $h_P(w)$ is a max for any $w \in \mathbb{R}^n$

$$P \subseteq \{x \in \mathbb{R}^n \mid w \cdot x \leq h_P(w)\} := \underline{\text{half space}}$$

with <u>supporting hyperplane</u>

$$\{x \in \mathbb{R}^n \mid w \cdot x = h_P(w)\}$$

Note the face

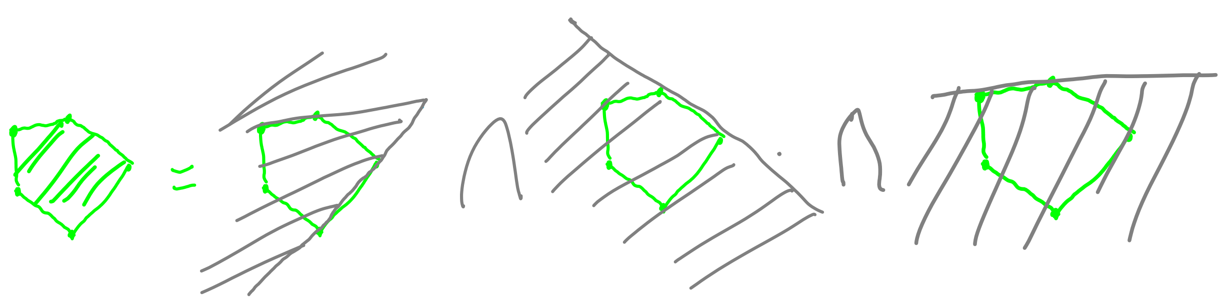
$$P_w = P \cap \text{supporting hyperplane corresponding to } w \in \mathbb{R}^n$$

P is closed, convex meaning its the intersection of half-space which contain P

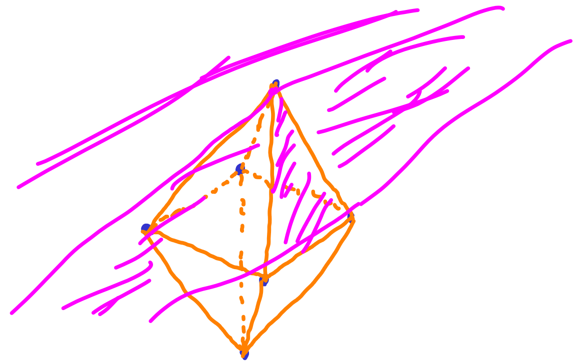
$$P = \bigcap_{w \in \mathbb{R}^n} \{x \in \mathbb{R}^n \mid w \cdot x \leq h_P(w)\}$$

Ex

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 & 0 \\ 1 & 2 & 2 & 1 & 1 \end{bmatrix}$$

$P = \mathrm{conv}(A) =$ 

Ex $A = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$



$P = \operatorname{conv}(A)$

$P = \{ (x,y,z) \in \mathbb{R}^3 \mid |x| + |y| + |z| \leq 1 \}$

intersection of the 8-bounding half spaces

$H_{(\pm,\pm,\pm)} := \{ (x,y,z) \in \mathbb{R}^3 \mid \pm x \pm y \pm z \leq 1 \}$

each face is $H_{(\pm,\pm,\pm)} \cap P$

Prop A polytope is the intersection of finitely many halfspaces, one for each facet of P.