Kushnirenko's Thm (Recap)

work $\mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$

Thm/ Fix some $r \times m$ integer matrix $A = [a_1, \dots, a_m]$

consider the sq. system

$$f_1(x) = f_2(x) = \cdots = f_r(x) = 0 \quad \text{in } \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}] \quad (*)$$

$$f_i = \sum c_{ij} x^{a_j}$$

$\uparrow$ support A

The system $(*)$ has at most $r!\,\mathrm{Vol}(\mathrm{conv}(A))$ solutions in $(\mathbb{C}^*)^r$ for any choice of $c_{i,j} \in \mathbb{C}^*$

when c_{ij} is generic, then $(*)$ has exactly $r!\,\mathrm{vol}(\mathrm{conv}(A))$ sols.

$$(\mathbb{C}^*)^r \longmapsto \mathbb{P}^{m-1}$$

$$\varphi_A : x \longmapsto (x^{a_1}, \dots, x^{a_m})$$

Lemma/ Set $L = V(\Lambda_1) \cap \cdots \cap V(\Lambda_r)$

for $\Lambda_j = \sum c_{ij} z_i \in \mathbb{C}[z_1, \dots, z_m]$

The solutions to $(*)$ are.

$$\varphi_A^{-1}(L) = \varphi_A^{-1}(L \cap \varphi_A((\mathbb{C}^*)^n))$$

Recall For $X \subseteq \mathbb{P}^n$ $\quad \dim(X) = r$

$$\deg(X) = \# X \cap L \quad \text{where}$$

$$L = V(\Lambda_1, \dots, \Lambda_r)$$

for Λ_i are general linear poly.

$\therefore \quad \deg(X_{A^+}) = A(L \cap \varphi((\mathbb{C}^*)^r))$

$\therefore$ By lemma

$\# \text{ sol to } (*) = \deg(X_{A^+})$

Summary

Lemma | If $\mathrm{Aff}_{\mathbb{Z}} A = \mathbb{Z}^m$ then the # of solutions to $(*)$ is given by $\deg(X_{A^+})$.

Goal | To prove Kushnirenko's thm when $\mathrm{Aff}_{\mathbb{Z}} A = \mathbb{Z}^m$ we will show

$$\deg(X_{A^+}) = r!\, \mathrm{vol}(\mathrm{conv}(A)).$$

Hilbert polynomial

Let I be an ideal in $k_{<}[x_0, \dots, x_n]$

$S_<(I)$ = set of all monomials $x^b \notin \mathrm{in}_<(I)$

standard monomials of I with respect to $<$

Thm | The set $S_<(I)$ of standard monomials is a basis for the k-vector space $k[x_0, \dots, x_n]/I = R/I$.

Proof | The image of $S_<(I)$ in R/I is linearly independent, since if $f \neq 0$ and

$f = 0$ in $R/I \Rightarrow f \in I$

$\Rightarrow f$ has at least $\mathrm{in}_<(f)$ not in $S_<(I)$

$\therefore$ any polynomial

$$\sum c_\alpha x^\alpha \quad \text{s.t.} \quad x^\alpha \in S_<(I)$$

cannot be zero in R/I

$\therefore$ $x^\alpha \in S_<(I)$ are lin. ind.

Now prove $S_<(I)$ span R/I. Suppose they do not

$\Rightarrow$ $\exists$ $x^c \neq \sum c_\alpha x^\alpha$ mod I for $x^\alpha \in S_<(I)$

Assume x^c is the minimal such monomial

by assumption $x^c \notin S_<(I) \Rightarrow x^c \in in_<(I)$

$\Rightarrow$ $\exists$ $h \in I$ s.t. $in_<(h) = x^c$

Each other monomial in h is smaller w.r.t $<$ and $\therefore$ is in the k-span of $S_<(I)$ mod I.

and $h = 0$ mod I

$\hookrightarrow$ $h = x^c$ + terms in k-span of $S_<(I) = 0$ mod I

This is a contradiction $\blacksquare$

Treat $\mathbb{C}[x_0, \dots, x_n]$ as a <u>graded ring</u>

$$\mathbb{C}[x_0, \dots, x_n] = \bigoplus_{d \geq 0} \underbrace{\mathbb{C}_d[x_0, \dots, x_n]}$$

all polys in $\mathbb{C}[x_0, \dots, x_n]$ which have degree d.

Standard monomials of degree d form a basis for $\mathbb{C}_d[x_0, \dots, x_n]$

$<$ = needs to be "degree compatible"
$\deg(x^a) < \deg(x^b) \Rightarrow x^a < x^b$

<u>Def</u> (Hilbert function) let $I \subseteq k[x_0, \dots, x_n]$ be a homogeneous ideal

The <u>Hilbert function</u> h_I takes $\mathbb{N} \to \mathbb{N}$

$$h_I(d) = \text{the number of monomials of degree } d \text{ not belonging to } in_<(I)$$
$$= \# \text{ standard monomials of degree} = d$$
$$= \dim_K \left(k[x_0, \ldots, x_n]/I \right)_d$$

Def] (Hilbert series) Let $I \subseteq k[x_1, \ldots, x_n]$, fix a formal variable d. The Hilbert series is

$$HS_I(d) = \sum_{q=0}^{\infty} h_I(q)\, d^q$$

Ex] $I = \{0\}$, i.e. count all monomials in $k[x_1, \ldots, x_n]$ of a fixed degree

$$HS_{\{0\}}(d) = \frac{1}{(1-d)^n} = \sum_{q=0}^{\infty} \binom{n+q-1}{n-1} z^q$$

Note that

$$\binom{n+q-1}{n-1} = \frac{(q+1)\cdots(q+n-1)}{(n-1)!}$$ is a poly in q.

Thm] The Hilbert series of I is

$$HS_I(z) = \frac{K_I(z)}{(1-z)^n}$$

where $K_I(z)$ is a polynomial with integer coefficients and $K_I(0) = 1$. Further $\exists$ a poly, HP, called the <u>Hilbert polynomial</u> of I s.t.

$HP(q) = h_I(q)$ for sufficiently large q

Thm 1 (Dimension and Degree) Let I be a ideal and write

$$HP_I(q) = \frac{g}{(d-1)!} q^{d-1} + \text{lower terms}.$$

If $HP \neq 0$ the dimension of I is $d = \dim(V(I))$

and $\deg(I) = \deg(V(I)) = g$

If $HP = 0$ we say I is zero dimensional, $\dim(V(I)) = 0$

$$\deg(I) = \deg(V(I)) = \dim_k\left(k[x_1, \dots, x_n]/I\right)$$

Homogeneous ideal $I_{A^+} \subseteq \mathbb{C}[x_1, \dots, x_m]$

Goal: Compute $\deg(X_{A^+})$

$\Updownarrow$

Find the Hilbert polynomial of the homogeneous toric ideal I_{A^+} $\left(A = [a_1, \dots, a_m],\ A^+ = \begin{bmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_m \end{bmatrix}\right)$

— Compute $h_{I_{A^+}}(d)$ $\forall d$

The homogeneous coordinate ring of X_{A^+} is $R = \mathbb{C}[x_0, \dots, x_n]/I_{A^+}$

By corollary 1.3 $R \cong \mathbb{C}[\mathbb{N}A^+]$

$\uparrow$ semi-group algebra

$\quad\ \ \|$

Complex linear combinations of elements of $\mathbb{N}A^+$.

$$\varphi_{A^+}(t,x) \longmapsto [\,t\, x^{a_1} : \cdots : t\, x^{a_n}\,]$$

$\uparrow$ homogenity parameter / scaling

$\therefore$ In $\mathbb{C}[z_1,\dots,z_m]/I_{A^+}$ the vector

$(d,a) \in \mathbb{N}A^+$ will correspond to polynomials

of degree d via pullback map

$$\varphi^*_{A^+} : \mathbb{C}[z_1,\dots,z_m] \to \mathbb{C}[t^{\pm}, x_1^{\pm}, \dots, x_n^{\pm}]$$

$$z_i \longmapsto t\, x^{a_i}$$

$f(z)$ homogeneous of degree d

$z_1^{d_1} \cdots z_m^{d_m}$, $d_1 + \cdots + d_m = d$ $\qquad \hat{a} = d_1 \cdot a_1 + \cdots + d_n a_n \in \mathbb{N}A$

$$\varphi^*_{A^+}(z_1^{d_1} \cdots z_m^{d_m}) = t^{\overbrace{d_1+\cdots+d_m}^{=d}} (x^{a_1})^{d_1} \cdots (x^{a_n})^{d_n}$$

$$(d, \hat{a})$$

i.e. via the pullback map

$$\left(\mathbb{C}[z_1,\dots,z_m] / I_{A^+} \right)_d \xrightarrow{\sim} \mathbb{C}_d[\mathbb{N}A^+]$$

$\uparrow$ elements of $\mathbb{C}[\mathbb{N}A^+]$ with first coordinate d

$$\mathbb{C}[\mathbb{N}A^+] = \bigoplus_d \mathbb{C}_d[\mathbb{N}A^+]$$

and the set $\{(d,a) \in \mathbb{N}A^+\}$ is a basis for

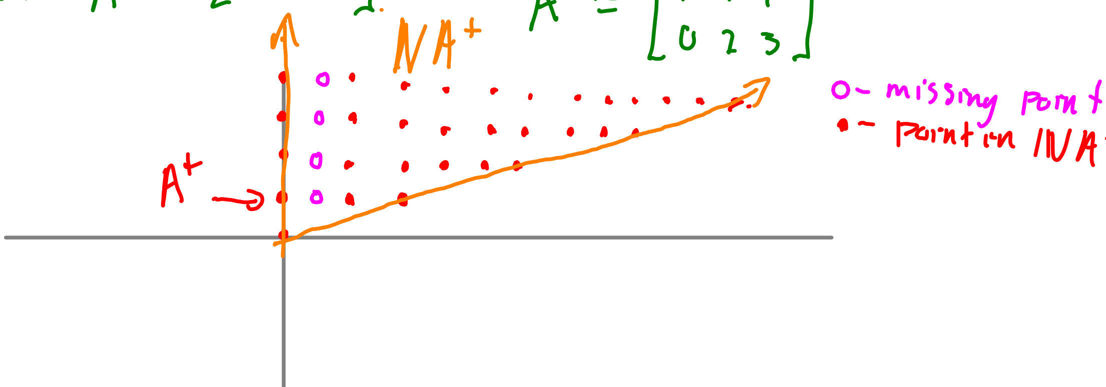
$\mathbb{C}_d[\mathbb{N}A^+]$.

Note that since each vector in A^+ has 1 as its first coordinate

$$dA^+ := \{(d,a) \in \mathbb{N}A^+\} \longleftrightarrow^{\text{bijection}} \overbrace{\mathbb{N}A^+ \cap \underbrace{d\,\mathrm{conv}(A^+)}_{\text{d-fold sums of vectors in conv}(A^+)}}^{\text{set of d-fold sums of vectors in } A^+}$$

$\therefore$ $h_{I_{A^+}}(d) = |dA^+|$ = # of vectors in $\mathbb{N}A^+$ which are d-fold sums of vectors in A^+.

Example Let $A = [0\ 2\ 3]$. $A^+ = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}$



So the Hilbert function of I_{A^+} has values

1, 3, 6, 9, 12, 15, ... for d = 0, 1, 2, ...

$\therefore$ $hP_{I_{A^+}}(d) = 3d$ is the hilbert polynomial

the "large" value for which hP agrees with the hilbert function is $d \geq 1$ here.

$h_{I_{A^+}}(d) = |dA^+|$ for any d

Note that since $dA^+ = \{(d,a) \in \mathbb{N}A^+\}$ = Set of d-fold sums of vec's in A^+

$\updownarrow$ bijection

dA = Set of d-fold sums of vectors in A

$$dA = d\,\mathrm{conv}(A) \cap \mathbb{N}A$$

Clearly

$$|d\,\mathrm{conv}(A) \cap \mathbb{N}A| \leq |d\,\mathrm{conv}(A) \cap \mathbb{Z}^r|$$

$\therefore$

$$h_{I_{A^*}}(d) \leq |d\,\mathrm{conv}(A) \cap \mathbb{Z}^r|$$

<u>Thm</u> (Ehrhart) Let $P \subseteq \mathbb{Z}^r$ be a convex polytope with integer vertices. The <u>Ehrhart function</u> is a map

$$E_P : \mathbb{N} \longrightarrow \mathbb{N}$$
$$d \longmapsto |dP \cap \mathbb{Z}^r|$$

$\uparrow$ Number of int points which are d-fold sums of vectors in P.

$E_P(d)$ is a polynomial in d. Further

$$\deg(E_P(d)) = \dim(P) \quad \text{and} \quad LC(E_P(d)) = \mathrm{Vol}(P)$$

$\uparrow$ lead coefficient.

<u>Ex</u> $P = \mathrm{Conv}([0, 3]) = [0, 3]$

$\uparrow$ there are 4 integer points in $[0,3]$

7 points which are two fold sums, ...

$$E_P(d) = 3d + 1$$

Now take $P = \mathrm{conv}(A)$ since $dA \subseteq \mathrm{conv}(A) \cap \mathbb{Z}^r$

$$h_{I_{A^+}}(d) \leq E_{\mathrm{conv}(A)}(d) \qquad \forall d$$

Goal : find $hP_{I_{A^+}}(d)$

Have an upper bound

Subgoal : Find a lower bound for $h_{I_{A^+}}(d)$ $\forall d$

Idea : Use inclusion of semi-groups

Set

$$S_A = \mathbb{R}_{\geq 0} A^+ \cap \mathbb{Z}^{n+1} =$$ semigroup of all integer points in the positive span of A^+

The inequality $h_{I_{A^+}}(d) \leq E_{conv}(d)$ arose from

$$\mathbb{N}A^+ \subseteq S_A$$

by looking at points with 1^{st} coord d.

Now we will construct a vector $v \in \mathbb{N}A^+$ and show that

$$v + S_A \subseteq \mathbb{N}A^+$$

Set $\mathcal{B} \subseteq S_A$ to be

$\mathcal{B}$ = set of points $b \in \mathbb{Z}^{n+1}$ s.t. $b = \sum_{i=1}^{m} \lambda_i (1, a_i)$

where $\lambda_i \in \mathbb{Q}$, $0 \leq \lambda_i < 1$

Ex For $A = [0, 2, 3]$ $\qquad \begin{bmatrix} 1 \\ 3 \end{bmatrix} \notin \mathcal{B}$

$$\mathcal{B} = \text{cols of } \begin{bmatrix} 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$