Aside | Normality of a Variety

- An Integral Domain S is called **Normal** if the roots in $\mathrm{frac}(S)$ for every monic poly in $S[x]$ are already in S.

- A variet $X = V(I) \subseteq k^n$ is normal if $k[x_1,\dots,x_n]/I$ is normal (assume I is radical)

Def | A lattice polytope $P \subseteq \mathbb{R}^m$ ($= \mathrm{conv}(A)$) is normal if for any $k \in \mathbb{Z}$ $u \in kP \cap \mathbb{Z}^m$, $\exists\, v_1, \dots, v_k \in P \cap \mathbb{Z}^m$ s.t $u = \sum v_i$. In this case $X_A \subseteq \mathbb{P}^n$ is (Projective ly) normal i.e $\mathbb{C}[x_0,\dots,x_n]/I_A$ is a normal ring.

Remark: Face of polytope uses same def as face of cone except the sum of coeeficents of the affine combo $= 1$.

Thm | The torus orbits in $X_A \subseteq \mathbb{P}^n$ are in bijection with the faces of the polytop $P = \mathrm{conv}(A)$. The orbit of a face F is

$$\sigma(F) = \{ y \in X_A \mid y_i \neq 0 \text{ whenever } a_i \in F \}$$

$$\overline{\sigma(F)} = \{ (y_0, \dots, y_n) \mid y_i = x^{a_i} \text{ if } a_i \in F,\ y_i = 0 \text{ otherwise} \}$$

$$\dim(\overline{\sigma(F)}) = \dim(F)$$

Inclusion of orbit closures in X_A corespnds to inclusion of faces in P.

Proof Apply affine version to the affine cone

- $\hat{X}_A = V(I_A) \subseteq \mathbb{C}^{n+1}$

$\Rightarrow$ the orbits of $\hat{X}_A$ correspond to faces of the cone C over the polytope P

Note $\dim(C) = \dim(P) + 1$
$$\dim(P)+1 = \#\text{ of rows of } A \quad (\text{if } A \text{ has full rank})$$

Each i-face of P corresponds to an $i+1$ face of C

- the face $\{0\}$ of C is not in P

  but this corresponds to $0 \in \hat{X}_A$, which is not in X_A. ∎

Suppose A is $(m+1)\times(n+1)$ matrix which lies in an affine hyperplane (so I_A is homogeneous)

Take $u, v \in \mathbb{N}^{m+1}$. $u \neq v \neq 0$, $Au = Av$

$\Rightarrow$ $z^u - z^v \in I_A$ is homogeneous

$$\deg(z^u) = \deg(z^v) = d = \sum u_i = \sum v_i$$

Let $\lambda_i = \frac{1}{d} u_i$, $\mu_i = \frac{1}{d} v_i \in \mathbb{Q}$

$$\sum a_i \lambda_i = \sum a_i \mu_i \qquad \lambda_i, \mu_i \geq 0 \qquad \sum \lambda_i = \sum \mu_i = 1$$

$\therefore$ The point

$$\alpha = \sum a_j \lambda_j = \sum a_j \mu_j$$ is a point $\operatorname{conv}(A)$

having two distinct representations as a combination of points in A.

Def) For $v \in \mathbb{Z}^{m+1}$, $\operatorname{supp}(v)$ = indices with non-zero entries
$\operatorname{supp}(v)$: support of v

Suppose $Aw = 0 \Rightarrow z^{w^+} - z^{w^-} \in I_A$

$\operatorname{supp}(w^-) \cap \operatorname{supp}(w^+) = \emptyset$ by def w^-, w^+

If take $u = w^+$, $v = w^-$

$$\alpha = \sum_{i \in \operatorname{supp}(w^+)} a_i \lambda_i = \sum_{i \in \operatorname{supp}(w^-)} a_i \mu_i$$

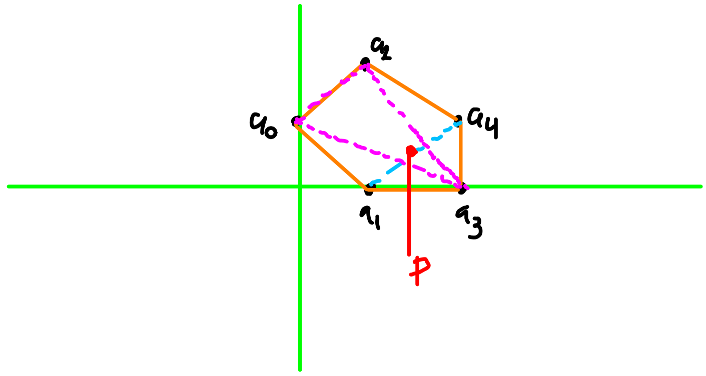
α: a rational point common to the convex hulls of two disjoint subsets of A.

Ex) $$A = \begin{array}{c} \begin{array}{ccccc} a_0 & \cdots & & & a_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 2 & 2 \\ 1 & 0 & 2 & 0 & 1 \end{pmatrix} \end{array}, \quad A^+ = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 1 & 0 & 2 & 0 & 1 \end{pmatrix}$$

The point $P = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}$ lies in $\operatorname{conv}\left(\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right) = \operatorname{conv}(a_1, a_4)$ and $\operatorname{conv}\begin{array}{c} \begin{array}{ccc} a_0 & a_2 & a_3 \end{array} \\ \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$

$$P = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2} a_1 + \frac{1}{2} a_4 = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \frac{1}{6} a_0 + \frac{1}{6} a_2 + \frac{2}{3} a_3 = \frac{1}{6}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$



Corresponds to the binomial $x_1^3 x_4^3 - x_0 x_2 x_3^4 \in I_{A^+}$

Another Relation with Polytopes/toric varieties:

Consider the map $\mu_A : \mathbb{P}^n \longrightarrow \mathrm{conv}(A)$

$$[z_0 : \cdots : z_n] \longmapsto \frac{\sum a_i |z_i|}{\sum |z_i|}$$

alg. moment map

Thm The map $\mu_A : X_{A^+} \to \mathrm{conv}(A)$ is surjective.
If F is a face of $P = \mathrm{conv}(A)$

$$\mu_A^{-1}(F) = X_{F_A}, \quad \text{where } F_A = F \cap A$$

includes interior points and vertices of F

The map μ_A remains surjective when restricted to

$X_{A^+}(\mathbb{R})$ = all real points in X_{A^+}

and when restricted to

$$X_{A^+}(\mathbb{R}_{\geq 0}) := \{ [x_0 : \cdots : x_n] \in X_{A^+} \mid x_i \geq 0 \ \forall i \}$$

The map $\mu_A : X_{A}(\mathbb{R}_{\geq 0}) \to \mathrm{conv}(A)$ is a homeomorphism.

Kushnirenko's Thm

Work in the Laurent polynomial Ring $\mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$

Def | The support of a Laurent poly

$$f = \sum c_a x^a$$

finitely many $c_a \neq 0$

is $\mathrm{supp}(f) =$ set of exponents $\{a_1, \dots, a_\ell\}$ s.t. $c_i \neq 0$
= the set of monomials $\{x^{a_1}, \dots, x^{a_\ell}\}$
= the columns of $A = [a_1, \dots, a_\ell]$

Def | (Newton Polytope)
For $f \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ the Newton Polytope of f is

$$\mathrm{Newt}(f) := \mathrm{conv}(\mathrm{supp}(f))$$

convex hull of the support

Thm (Kushnirenko, 1976) Fix a $r \times m$ integer matrix $A = [a_1, \dots, a_m]$ and consider the square poly system

(*) $f_1(x) = f_2(x) = \cdots = f_r(x) = 0$ in $\mathbb{C}[x_1^{\pm}, \ldots, x_r^{\pm}]$

with $f_i(x) = \sum_{j=1}^{m} c_{i,j} x^{a_j}$.

The system (*) has at most $r!\,\mathrm{vol}(\mathrm{conv}(A))$ many solutions in $(\mathbb{C}^*)^r$ for any choice of $c_{i,j} \in \mathbb{C}^*$

Further for $C = [c_{ij}]$ chosen from a Zariski dense subset of $(\mathbb{C}^*)^{r \times m}$ the system (*) has exactly $r!\,\mathrm{vol}(\mathrm{conv}(A))$ solutions, each of which is isolated and has mult. one.

Aside | Exists some Zariski open set $U \subseteq (\mathbb{C}^*)^{r \times m}$ s.t. $\#\text{sols} = r!\,\mathrm{vol}(\mathrm{conv}(A))$ for all $C \in U$, i.e. $U = (\mathbb{C}^*)^{r \times m} - V$

V a subvariety

A key step in this proof will be the Lemma

Lemma | when $\mathbb{Z}A = \mathbb{Z}^r$ then $\deg(X_{A^+}) = \underbrace{r!\,\mathrm{vol}(\mathrm{conv}(A))}_{\text{Normalized volume of } P}$

$P = \mathrm{conv}(A)$

Fix a $r \times m$ integer matrix A, with full rank.

Consider the injectivity of

$$\varphi_{A^+} : (\mathbb{C}^*)^{r+1} \longrightarrow \mathbb{P}^{m-1}$$

<u>Def</u> (Affine hull/span) The <u>affine hull</u> or <u>affine span</u> of A is

$$\mathrm{Aff}(A) := \left\{ \sum_{i=1}^{m} \lambda_i a_i \;\middle|\; \sum_{i=1}^{m} \lambda_i = 1 \right\}$$

"weakening" of convex hull condition to allow $\lambda_i < 0$.

For any $a \in \{a_1, .., a_m\}$

$$\mathrm{Aff}(A) = a + \mathbb{R}\{ b - a \mid b \in \{a_1, .., a_m\} \}$$

the <u>integral affine span</u> is

$$\mathrm{Aff}_{\mathbb{Z}}(A) = a + \mathbb{Z}\{ b - a \mid b \in \{a_1, .., a_m\} \}$$

$$\varphi_{A^+} : \mathbb{C}^* \times (\mathbb{C}^*)^r \longrightarrow \mathbb{P}^{m-1}$$

$$(t, w) \longmapsto [t w^{a_1} : \dots : t w^{a_m}]$$

and define

$$\varphi_A : (\mathbb{C}^*)^r \longrightarrow \mathbb{P}^{m-1}$$

$$w \longmapsto [w^{a_1} : \dots : w^{a_m}]$$

$$\varphi_A = \varphi_{A^+} \big|_{\{1\} \times (\mathbb{C}^*)^r}$$

i.e. restricting the domain to $t = 1$

In Ex. 6 you will show that

φ_A is injective iff $\mathrm{Aff}_{\mathbb{Z}}(A) = \mathbb{Z}^n$

Remark/ Note that if $0 \in A$ (i.e. A has a zero column) then

$$\mathrm{Aff}_{\mathbb{Z}}(A) = 0 + \mathbb{Z}\{b - 0 \mid b \in A\} = \mathbb{Z}A$$

In context of Kushnirenko's Thm we may assume WLOG that $0 \in A$ since for a Laurent poly f with $\mathrm{supp}(f) = \{a_1, \dots, a_m\}$ the zero set of f and $\underbrace{\tilde{f} = x^{-a_i} f}_{\tilde{f} \text{ has a constant term}}$ is identical in $(\mathbb{C}^*)^n$

$\tilde{f} = x^{a_1} f$

$\mathrm{Supp}(\tilde{f}) = \{0, a_2, \dots, a_m\}$

When proving Kushnirenko's theorem we may assume $0 \in A$ and φ_A is injective iff $\mathbb{Z}A = \mathbb{Z}^n$.

Note since $0 \in A$, say $a_1 = 0$ then

$$\varphi_A : (1, w) \longmapsto \underset{\substack{\shortparallel \\ [1 : w^{a_2} : \dots : w^{a_m}]}}{[w^0 : w^{a_2} : \dots : w^{a_m}]} \in \mathbb{P}^{m-1}$$

i.e. $\varphi_A((\mathbb{C}^*)^r) \subseteq$ standard affine patch of $\mathbb{P}^{m-1}$ where $z_1 \neq 0$

Consider a linear form

$$\Lambda = \sum_{i=1}^{m} c_i z_i \quad \in \mathbb{C}[z_1, .., z_m]$$

Its pull back along φ_A is a polynomial in $\mathbb{C}[x_1, .., x_r]$ with support A:

$$\varphi_A^*(\Lambda) = \sum_{i=1}^{m} c_i x^{a_i}$$

$\therefore$ The square poly system $(*)$ with support A is the pull back of a system of r linear forms in $\mathbb{C}[z_1, .., z_m]$

Since r distinct homogenous linear forms define r distict hyperplanes in $\mathbb{P}^{m-1}$ $\Rightarrow$ they intersect in a space of codimension r

Set $L = V(\Lambda_1) \cap \cdots \cap V(\Lambda_r)$ for distict Λ_i

Lemma/ The solution set to the system $(*)$ with support A is

$$\varphi_A^{-1}(L) = \varphi_A^{-1}(L \cap \varphi_A((\mathbb{C}^*)^r))$$

Example/ Consider

$$f = x^2y + 2xy^2 - 1 + xy \quad , \quad g = x^2y - xy^2 + 2 - xy$$

The system $(*)$ in this case is $f = g = 0$

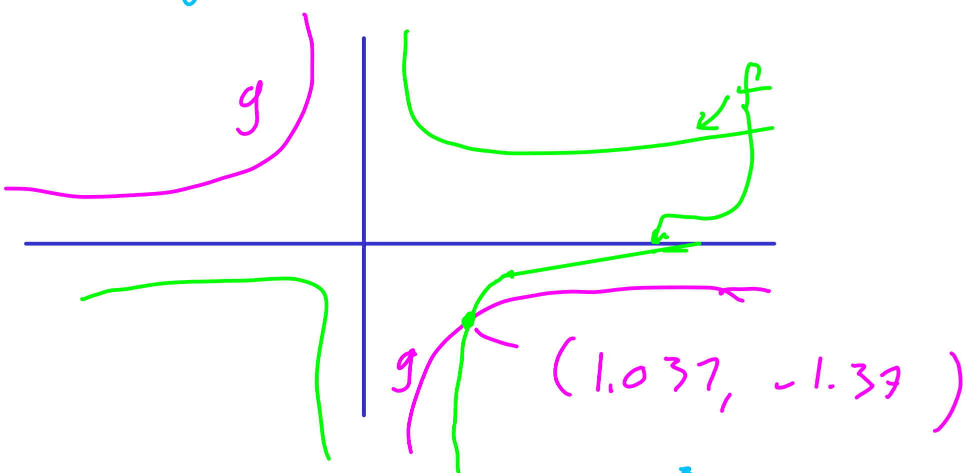
$$\mathrm{Supp}(f) = \mathrm{Supp}(g) = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 \end{pmatrix} := A$$

$$\varphi_A : (x,y) \longmapsto [x^2y : xy^2 : 1 : xy] \in \mathbb{P}^3$$

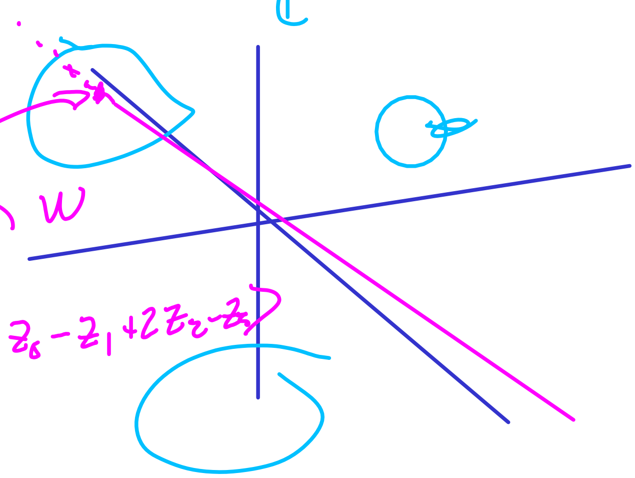
$$\varphi_A((\mathbb{C}^*)^2) = \{[z_0 : z_1 : z_2 : 1] \mid z_2^3 - z_0 z_1 = 0\} \subseteq \mathbb{P}^3$$

$$\overline{\varphi_{A^+}((\mathbb{C}^*)^2)} = V(z_2^3 - z_0 z_1 z_3) \subseteq \mathbb{P}^3$$

$f = g = 0$ has one real sol



$\mathbb{C}^3$



real point in $V(\ell) \cap W$

$$\ell = \langle z_0 + 2z_1 - z_2 + z_3,\ z_0 - z_1 + 2z_2 - z_3 \rangle$$

$$W = V(z_2^3 - z_0 z_1) \subseteq \mathbb{C}^3$$