Projective Toric Varieties

Take $A = [a_0, \dots, a_m]$. We say that the set $\{a_0, \dots, a_m\}$ lives in an affine hyperplane if $\exists\, w \in \mathbb{Z}^n$

$$w \cdot a = w \cdot b \quad \forall\, a, b \in \{a_0, \dots, a_m\}$$

Set $c = w \cdot a_i \ \forall i$ (The affine hyperplane $w \cdot x - c$ in $\mathbb{Q}[x_1, \dots, x_n]$)

Lemma / If $A = \{a_0, \dots, a_m\}$ lie in an affine hyperplane, then I_A is a homogeneous ideal and defines the projective toric variety $X_A = V(I_A) \subseteq \mathbb{P}^m$.

Proof / Suppose $z^u - z^v \in I_A$ (want to show this is homogeneous), then $Au = Av$, it is enough to show $\deg(z^u) = \deg(z^v)$

Since $Au = Av$, $u \in \mathbb{Z}^n$, then $w \cdot Au = w \cdot Av$ (dot prod; w defines affine plane)

$$w \cdot Au = [w \cdot a_0, \dots, w \cdot a_m] \cdot u = [w \cdot a_0, \dots, w \cdot a_m] \cdot v$$

$$= [c, \dots, c] \cdot u = [c, \dots, c] \cdot v$$

$$\cancel{c}(1, \dots, 1) \cdot u = \cancel{c}(1, \dots, 1) \cdot v \Longrightarrow u \cdot (1, \dots, 1) = v \cdot (1, \dots, 1)$$

$$|u| = |v|$$

$$\deg(z^u) = \deg(z^v)$$

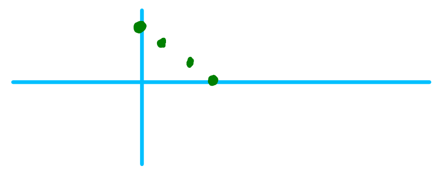
$\therefore\ z^u - z^v$ is homogeneous

∴ $I_A = \langle z^u - z^v \mid Au = Av \rangle$ is a homogeneous ideal

∴ X_A is a projective variety.

<u>Example</u>

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} = [a_0, \dots, a_3]$$



Note $a_i \cdot (1,1) = 3 \quad \forall i$ ∴ a_i lie in an affine hyperplane with $w = (1,1)$, $c = 3$

$$\varphi_A(w,z) = (w^3, w^2 z, w z^2, z^3) \in \mathbb{C}^4$$

Via elimination

$$I_A = \langle -x_2^2 + x_1 x_3,\ -x_1^2 + x_0 x_2,\ -x_1 x_2 + x_0 x_3 \rangle$$

vectors in $\ker(A)$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$b_1 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad b_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$A \cdot b_i = 0, \quad \ker_{\mathbb{Z}}(A) = \mathrm{span}_{\mathbb{Z}}(b_1, b_2, b_3)$$

The gcd of the 2x2 minors of A = 3

$\Rightarrow$ $\mathrm{rank}(\mathbb{Z}A) = 2$, $\quad |\mathbb{Z}^2 : \mathbb{Z}A| = 3$

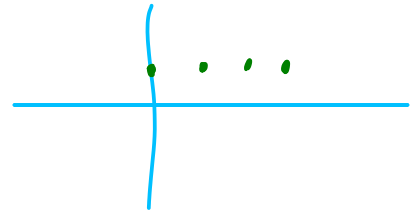
If we take

$\tilde{A}$ = generator of ker of B

where B = integer matrix s.t. $A \cdot B = 0$

then $|\mathbb{Z}^2 : \mathbb{Z}\tilde{A}| = 1$

Doing more row op



$$A' = \begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 2 & 3\end{bmatrix}$$

$$\varphi_{A'}(w,z) = (w, wz, wz^2, wz^3)$$

$I_A = I_{A'}$

Given a matrix $A = [a_0,\dots,a_n]$ in $\mathbb{Z}^{m\times(n+1)}$

define the lift of A as

$$A^+ = [a_0^+,\dots,a_n^+] \quad \text{in } \mathbb{Z}^{(m+1)\times(n+1)}$$

$$= \begin{bmatrix}1 & 1 & \cdots & 1\\ a_{00} & \cdots & & a_{0n}\\ \vdots & & & \\ a_{m0} & & & a_{mn}\end{bmatrix}$$

$a = a_i$

$$a^+ = \begin{bmatrix}1\\ a_1\\ \vdots\\ a_m\end{bmatrix}$$

By construction A^+ is on an affine hyperplane

Ex) $A^+ = A' = \begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 2 & 3\end{bmatrix}$ was the lift of $A = [0\ 1\ 2\ 3]$

$$w\cdot(1,0) = w\cdot(1,1) = w(1,2) = w\cdot(1,3)$$

$w_1 =$ free, $w_2 = 0$

$w = (t,0)$ for any t

Note the map

$$\varphi_{A^+} : (\mathbb{C}^*)^{m+1} \longrightarrow \mathbb{C}^{n+1}$$

has a natural structure when $A = [a_0,\dots,a_n]$

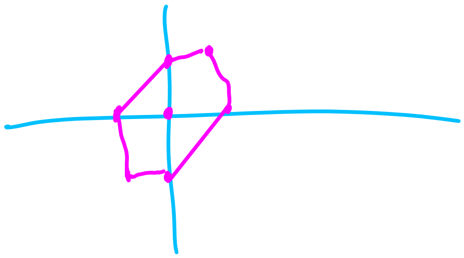
$$\varphi_{A^+} : \mathbb{C}^* \times (\mathbb{C}^*)^m \longrightarrow \mathbb{P}^n$$

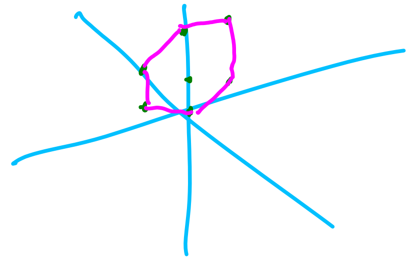
$$\underbrace{(t, z_1,\dots,z_m)}_{(t,z)} \longmapsto [t z^{a_0} : \cdots : t z^{a_n}]$$

$$\varphi_{A^+}(t_1, z) \sim \varphi_{A^+}(t_2, z) \quad \forall\, t_1, t_2 \in \mathbb{C}^*$$

Example]

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & -1 \end{bmatrix}$$





$$A^+ = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & -1 \end{bmatrix}$$

Polytopes]

Given an $m \times (n+1)$ integer matrix $A = [a_0, \dots, a_n]$ define the convex hull of A as

$$\mathrm{Conv}(A) = \left\{ \sum_{i=0}^{n} a_i \lambda_i \;\middle|\; \sum_{i=0}^{n} \lambda_i = 1 \text{ and } \lambda_i \geq 0,\ i = 0, \dots, n \right\}$$

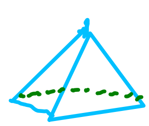
convex combination

For an int. mat A we call

$$P := \mathrm{conv}(A) \quad \text{a (lattice) polytope}$$

Ex] $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$P = \mathrm{conv}(A) =$ triangular pyramid

$=$ 

Polytopes, cones, faces, orbit cone correspondence

Def A <u>convex polyhedral cone</u> C in $\mathbb{R}^m$ is defined by vectors $\{v_1, \dots, v_k\}$, $v_i \in \mathbb{R}^m$ is

$$C = \left\{ \sum_{i=1}^{k} \lambda_i v_i \;\middle|\; \lambda_i \geq 0 \right\}$$

If $v_i \in \mathbb{Q}^m$ $\forall i$ we say C is a <u>rational</u> convex polyhedral cone.

Def A face F of a cone $C \subseteq \mathbb{R}^m$ is a subset of the form

$$F = \{ c \in C \mid l(c) = 0 \}$$

where l is a linear polynomial s.t. $l(c) \geq 0$ $\forall c \in C$

The <u>dimension</u> of a face F of a cone is the dimension of the smallest linear space which contains F

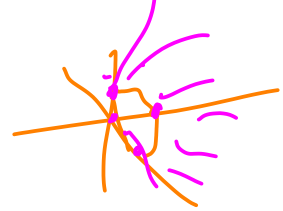
If $\dim(F) = m - 1 = \dim(C) - 1$ then we call F a <u>facet</u>. The choice of l is unique up to constant

A facet lies in a hyperplane H_F, we call this hyperplane H_F the <u>supporting hyperplane</u>.

Note face defined by $l = 0$ is $F = C$. Any face F of C is also a cone.

Ex) $C = \mathbb{R}^m_{\geq 0}$ is a cone

$\uparrow$ non-negative orthant



It has 2^m faces ranging from apex $0 = (0, \dots, 0)$ to C itself. There are $\binom{m}{i}$ faces of dimension i

In particular there are $\binom{m}{m-1} = m$ Facets

The m facet of dim $m-1$ are given by the m linear polynomials $l_i(x) = x_i$

i.e. $\mathcal{F}_i = \{ c \in \mathbb{R}^m_{\geq 0} \mid c_i = 0 \} \cong \mathbb{R}^{m-1}_{\geq 0}$

Def) If C is a convex polyhedral cone in $\mathbb{R}^m$ then its f-vector is the vector

$$f(C) = (f_1(C), \dots, f_{m-1}(C))$$

$f_i(C)$ = # of faces of dimension i of C.

Ex) For $C = \mathbb{R}^r_{\geq 0}$, $f(C) = (5, 10, 10, 5)$

Recall the (affine) toric variety $X_A = \overline{\varphi_A((\mathbb{C}^*)^m)} \subseteq \mathbb{C}^n$ is the closure of the torus $T = \varphi_A((\mathbb{C}^*)^m)$ in $\mathbb{C}^n$

This torus T acts on itself (by coordinate wise mult.) e.g. for $t, w \in T$ $t \cdot w = (t_1 w_1, \dots, t_n w_n)$ is the action of t on w

Similarly, this extends to an action on $\mathbb{C}^n$, i.e. $t \in T$ $w \in \mathbb{C}^n$

$$\underset{\substack{\downarrow \\ t \text{ acts on } w}}{t} \cdot w = (t_1 w_1, \ldots, t_n w_n)$$

$\therefore$ The group T acts on X_A.

Recall for a group G acting on a set X the orbit of $x \in X$ is

$$G \cdot x = \{ g \cdot x \mid g \in G \}$$

For $x \in X_A$ $T = \varphi_A((\mathbb{C}^*)^m)$

$$T \cdot x = \{ (t_1 x_1, \ldots, t_n x_n) \mid t \in T \}$$

note $\tilde{x} \in T \cdot x$ then $Tx = T\tilde{x}$.

Let $A = [a_1, \ldots, a_n]$ be $m \times n$ matrix

let C = cone generated by columns $\{a_1, \ldots, a_n\}$

Thm: The torus orbits in X_A are in bijection with the faces of the cone C. The orbit $\sigma(F)$ corresponding to a face F of C is

$$\sigma(F) = \{ x \in X_A \mid x_i \neq 0 \text{ whenever } a_i \in F \}$$

$$\overline{\sigma(F)} = V(F)$$

$$\overline{\sigma(F)} = (x^{a_i} \mid a_i \in F, x \in \mathbb{C}^m)$$

$$= \left\{ (y(a_1), \dots, y(a_n)) \in \mathbb{C}^n \;\middle|\; y(a_i) = \begin{cases} 0 & \text{if } a_i \notin F \\ x^{a_i} & \text{for all } x \in \mathbb{C}^m, \text{ if } a_i \in F \end{cases} \right\}$$

we also have $\dim(\sigma(F)) = \dim(F)$ further

$$\overline{\sigma(F_1)} \subseteq \overline{\sigma(F_2)} \quad \text{iff} \quad F_1 \text{ is a face of } F_2$$

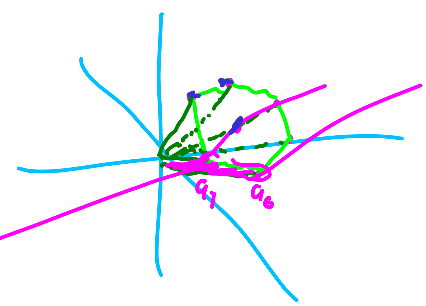
Example | Let

$$A = \begin{array}{c} \begin{array}{ccccccc} a_0 & & & & & & a_6 \end{array} \\ \begin{bmatrix} 2 & 2 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

interior point

$f(C) = (1, 6, 6, 1)$

C = Cone over $\{a_0, \dots, a_6\}$



Two dimesinal faces (6 of them)

$\mathrm{Codim}(F) = \dim(C) - \dim(F)$

$\{a_0, a_1\}$, $\{a_0, a_2\}$, $\{a_3, a_5\}$, $\{a_2, a_4\}$, $\{a_1, a_3\}$, $\{a_4, a_5\}$

$F = \mathbb{R}_{\geq 0}\{a_0, a_1\}$ corespond to

$$\overline{\sigma(F)} = \{(t, u, 0, 0, 0, 0, 0) \in X_A \mid t, u \in \mathbb{C}^*\}$$

The 1-dim faces = all vertices of hexagon

$\{a_0\}, \dots, \{a_5\}$

$F = \mathbb{R}_{\geq 0}\{a_0\}$ corespond

$$\overline{\sigma(F)} = \{(t, 0, 0, 0, 0, 0, 0) \in X_A \mid t \in \mathbb{C}^*\}$$

$F = \{0\} \quad \longleftrightarrow \quad$ the origin in $\mathbb{C}^7$

$F = C \quad \longleftrightarrow \quad T = \varphi_A((\mathbb{C}^*)^3) = X_A \cap (\mathbb{C}^*)^7$

~~Proof~~ Sketch | we will give a more complete proof later.

Idea

- Check that $\sigma(F)$ is actually a torus orbit (using the fact F is a face)

In particular, since $\mathbb{C}$ is alg closed, for each point $x \in \mathbb{C}^n$ we can construct a point $t \in T = \varphi_A((\mathbb{C}^*)^m)$ s.t

$t \cdot x = p_F$ where $(p_F)_i = 1$ if $a_i \in F$ and zero otherwise

(note t has no zero coordinates but p_F does, and x may or may not)

Then the orbit of x by T is some $\sigma(F)$

- Show that every point $x \in X_A$ is in in some $\sigma(F)$, Here you need to look at I_F

given by taking the colums of A which are in F