Recall : For an integer matrix $A = [a_1, \dots, a_m]$

$$I_A = \operatorname{Span}_{\mathbb{C}}(z^u - z^v \mid Au = Av)$$
$$= \langle z^u - z^v \mid Au = Av \rangle$$

To simplify this a bit lets re phrase

Suppose $Au = Av$, $u, v \in \mathbb{N}^m$

define vectors r, w^+, w^- in $\mathbb{N}^m$

$$r_i = \min(u_i, v_i)$$
$$w_i^+ = \max(u_i - v_i, 0)$$
$$w_i^- = \max(v_i - u_i, 0)$$

Note that $u = r + w^+$, $v = r + w^-$; $u - v = w^+ - w^-$

$$\therefore \quad \underbrace{z^r}_{z^r = \gcd(z^u, z^v)} (z^{w^+} - z^{w^-}) = z^u - z^v \in I_A$$

$$A(u-v) = A(w^+ - w^-) \quad \therefore \quad Aw^+ = Aw^-$$

$$\Rightarrow \quad z^{w^+} - z^{w^-} \in I_A$$

Check: $Au = Av$

Ex

$$A = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix} \end{array}, \quad u = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$x^u - x^v = x_1 x_3 x_5 - x_2 x_3 x_4 = \underbrace{x_3}_{x^r} \underbrace{(x_1 x_5 - x_2 x_4)}_{x^{w^+} - x^{w^-}}$$

$$w^+ = \begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix}, \quad w^- = \begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix}, \quad r = \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}$$

Def) For $u \in \mathbb{Z}^m$ define $u^+, u^- \in \mathbb{N}^m$ as

$$u_i^+ = \max(u_i, 0) = \begin{cases} u_i & \text{if } u_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$u_i^- = \max(-u_i, 0) = \begin{cases} |u_i| & \text{if } u_i < 0 \\ 0 & \text{otherwise} \end{cases}$$

Thm) $I_A = \langle x^{u^+} - x^{u^-} \mid \underbrace{Au = 0}_{u \in \ker_{\mathbb{Z}}(A)} \rangle$

Thm) Any Reduced Gröbner basis of I_A consists of binomials.

Question) By Hilbert's basis Thm we know $\exists$ some finite subset M of $\ker_{\mathbb{Z}}(A)$ s.t

$$I_A = \langle x^{u^+} - x^{u^-} \mid Au = 0 \rangle = \langle x^{u^+} - x^{u^-} \mid u \in \underbrace{M}_{\text{finite}} \rangle$$

What is this set?

A: Markov Basis.

Recall: A (undirected) graph $G = (V, E)$ is a collection of vertices V, and edges

$E =$ set of paired vertices, i.e. lines joining vertices

Def/ Let $A = [a_1, \dots, a_n]$, $b \in \mathbb{Z}^m$, The Fibre of A over b
length m vectors
as

$$\mathcal{F}_{A,b} := \{ u \in \mathbb{N}^n \mid Au = b \}$$

This is a finite set, it is the set
of integer points

Def/ A finite set $\mathcal{M}$ is a Markov basis for A
if the graph $\mathcal{F}(\mathcal{M}, b)$ is connected for all $b \in \mathbb{N}A$

$$\mathcal{F}(\mathcal{M}, b) = (V, E), \quad V = \mathcal{F}_{A,b}$$

E = there is an edge between u, v if
$u - v \in \pm \mathcal{M}$.

Def/ (Non-negative walk) Let $u, v \in \mathbb{N}^n$, a non-negative walk
from u to v on $\mathcal{M}$ is a list $u_0, \dots, u_s$
or
$u_0 \to u_1 \to \dots \to u_s$

s.t.
$u = u_0$, $u_s = v$, $u_i \in \mathbb{N}^n$, $u_{i-1} - u_i \in \pm \mathcal{M}$

Note: for such a walk to exist we need $u - v \in \mathbb{Z}\mathcal{M}$

Set $I_{\mathcal{M}} = \langle x^{m^+} - x^{m^-} \mid m \in \mathcal{M} \rangle$

Proposition/ There exists a non-negative walk between
$u, v \in \mathbb{N}^n$ on $\mathcal{M}$ if and only if

$$x^u - x^v \in I_{\mathcal{M}}.$$

Proof | $\Rightarrow$ If there exists a non-negative walk then

$$x^u - x^v = \underbrace{\underbrace{x^{u} - x^{u_1}}_{u=u_0} + \underbrace{x^{u_1} - x^{u_2}} + \underbrace{x^{u_2} + \cdots + (-x^{u_{s-1}})} + (x^{u_{s-1}}) - x^v}$$

$$x^{u_{\ell-1}} - x^{u_\ell} = \underbrace{x^r}_{\gcd} \underbrace{\left(x^{(u_{\ell-1}-u_\ell)^+} - x^{(u_{\ell-1}-u_\ell)^-} \right)}_{\in I_M}$$

$\Leftarrow$ Now suppose $x^u - x^v \in I_M$, then

$$x^u - x^v = \sum_{i=1}^{\ell} x^{r_i} \left(x^{m_i^+} - x^{m_i^-} \right)$$

Discussion Q for next week
why is it enough to take monomial coefficients

$\Rightarrow$ Must exist some index i_0 s.t

$$x^u = x^{r_{i_0}} \cdot x^{m_{i_0}^+}$$

First step of walk $u_0 = r_{i_0} + m_{i_0}^+$, $u_1 = r_{i_0} + m_{i_0}^-$

then either $-x^{r_{i_0} + m_{i_0}^-} = -x^v$

or it is canceled by some term $x^{r_{i_1} + m_{i_1}^+}$

In the second case we take the next step as $u_2 = r_{i_1} + m_{i_1}^-$

After finitely many steps we have our complete walk.

Thm | A finite set $M \subseteq \ker_{\mathbb{Z}}(A)$ is a Markov basis for A if and only if

$$I_A = I_M = \langle x^{m^+} - x^{m^-} \mid m \in M \rangle$$

Proof Since $M \subseteq \ker_{\mathbb{Z}}(A)$ and $I_A = \langle x^{u^+} - x^{u^-} \mid u \in \ker_{\mathbb{Z}}(A) \rangle$

then clearly $I_M \subseteq I_A$

Suppose M is a Markov basis, then if $x^u - x^v \in I_A$

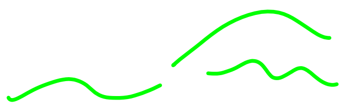
then $Au = Av$, and since M is a markov basis

$\exists$ a non-negative walk from u to v in M

$\therefore$ By Prop. $\Rightarrow$ $x^u - x^v \in I_M$

$\Rightarrow I_A \subseteq I_M$ ∎

Projective Varieties



Ussually $V = k^{n+1}$

$\uparrow$ some field.

Given a vector space V, $\dim(V) = n+1$,

The projective space on V is

$\mathbb{P}^n := \mathbb{P}(V)$ = The points are lines through the origin in V

written as $[a_0 : \dots : a_n] \in \mathbb{P}(V)$

for a line in V through $(a_0, \dots, a_n) \in V$.

Formally: $\mathbb{P}(V) = V - \{0\} / \sim$

= Set of equivilence classes $[v]$ for $v \in V - \{0\}$

where $v_1 \sim v_2$ iff $v_1 = \lambda v_2$ for some $\lambda \in k^*$

For $V = \mathbb{C}^{n+1}, \mathbb{R}^{n+1}$

$\mathbb{P}(V)$ is compact in the classical topology

$\mathbb{P}(V)$ is covered by (Zariski open) subsets

$$S_i = \{ a \in \mathbb{P}(V) \mid a_i \neq 0 \} = \{ [a_0 : \cdots : a_{i-1} : 1 : a_{i+1} : \cdots : a_n] \in \mathbb{P}(V) \}$$

To define varieties in $\mathbb{P}^n$ we need to restrict to homogeneous poly.

For an arbitrary $f \in K[x_0, \dots, x_n]$ we could have

$$P_1 \sim P_2 \text{ in } \mathbb{P}^n, \text{ but } f(P_1) \neq f(P_2)$$

$$\Big(\underbrace{P_1 \neq P_2}_{\text{in general}} \text{ in } K^{n+1} \Big)$$

A polynomial $f \in K[x_0, \dots, x_n]$ is called <u>homogeneous of degree d</u> if all monomials have the same degree

$$f = \sum c_a x^a, \text{ then } \deg(x^a) = |a| = \underset{\uparrow}{d} \text{ for all } c_a \neq 0$$

fixed integer d

Def Let $f_1, \dots, f_s$ be homogeneous polynomials, $I = \langle f_1, \dots, f_s \rangle$ then the associated <u>Projective Variety</u> is

$$V(f_1, \dots, f_s) = V(I) = \{ [a_0 : \cdots : a_n] \in \mathbb{P}^n \mid f_i(a_0, \dots, a_n) = 0 \ \forall i = 1, \dots, s \}$$

Note: If f is homogeneous of degree d

$$f(\lambda p_0, \dots, \lambda p_n) = \sum c_a (\lambda p_0)^{a_0} \cdots (\lambda p_n)^{a_n} = \sum c_a \lambda^d p^a$$

$$= \lambda^d f(p)$$

$$\therefore \text{ if } f(p) = 0 \Rightarrow f(\lambda p_0, \dots, \lambda p_n) = 0$$

The ideal I is called <u>homogeneous</u> if $\exists$ a generating set for I consisting of homogeneous polynomials.

E.g. $\langle x^2y - z^3, x-y \rangle$ is homogeneous

Ex) $I = \langle x + y^2, y \rangle$ is a homogeneous ideal

$\| \leftarrow$ a Gröbner basis of I

$\langle x, y \rangle$

Given a projective variety $X = V(I) \subseteq \mathbb{P}^n$ the <u>affine cone</u> $\hat{X}$ over X is the variety $V(I) \subseteq k^{n+1}$, i.e the solution set of the same eq.s in k^{n+1}.

<u>Def</u> (Degree and Dimension) Let $V \subseteq \mathbb{P}^n$, or $V \subseteq k^n$, k - alg. closed be an affine or projective variety.

Let $L(i) = H_1 \cap \cdots \cap H_i$ where H_i are general hyperplanes

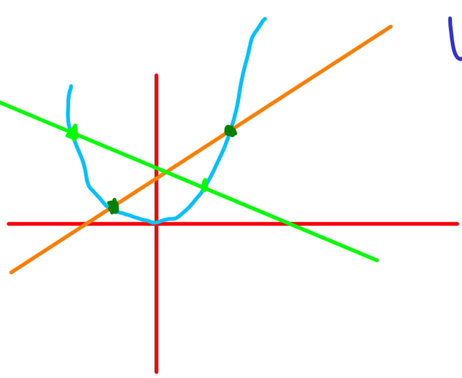
in k^n, $H_\ell = V(\lambda_1 x_1 + \cdots + \lambda_n x_n + \lambda_0)$ for general $\lambda_i \in k$

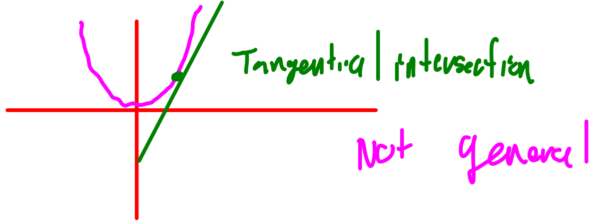
in $\mathbb{P}^n$ $H_\ell = (\lambda_0 x_0 + \cdots + \lambda_n x_n)$

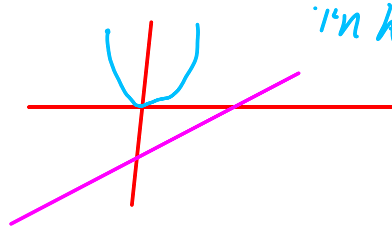
The <u>dimension</u> of V is the $m \in \mathbb{N}$ such that

$$V \cap L(m) = \text{a set of isolated points}$$

$$V \cap L(m+1) = \emptyset$$







If V has dimension m, then the degree of V is

$$\deg(V) := \#(V \cap L(m))$$

If $X \subseteq \mathbb{P}^n$ and $\hat{X} \subseteq k^{n+1}$ is its affine cone

$$\dim(X) = \dim(\hat{X}) - 1 \quad , \quad \deg(X) = \deg(\hat{X})$$

If $Y = V(I)$, $I \subseteq k[x_1, \dots, x_n]$, is any variety in k^n we may take its projective closure to obtain a variety $\overline{Y} = V(\overline{I}) \subseteq \mathbb{P}^n$ where

$$\overline{I} = \{ \underbrace{x_0^{\deg(g)} \cdot g\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)}_{\text{the homogenization of } g} \mid g \in I \}$$

Thm Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal. If $G = \{g_1, \dots, g_s\}$ is a reduced Gröbner basis of I (in Graded lex order), then

$$\overline{I} = \left(x_0^{\deg(g_1)} g_1\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right), \dots, x_0^{\deg(g_s)} g_s\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \right)$$

Cor If $Y \subseteq k^n$ then $\dim(\overline{Y}) = \dim(Y)$

$$\deg(\overline{Y}) = \deg(Y).$$

Projective toric varieties

Take $A = [a_0, \ldots, a_m]$ (column vectors in $\mathbb{Z}^n$). We say that the set $\{a_0, \ldots, a_m\}$ lies in an affine hyperplane if $\exists$ $w \in \mathbb{Z}^n$ with

$$w \cdot a = w \cdot b \quad \forall\, a, b \in \{a_1, \ldots, a_m\}$$

Set $c = w \cdot a_i$, The affine hyper-plane is defined by

$$w \cdot x - c = \underbrace{w_1 x_1 + \cdots + w_n x_n - c}_{\text{Hyperplane in } k^n,\ V(w \cdot x - c)} \in k[x_1, \ldots, x_n]$$

Lemma / If $A = \{a_0, \ldots, a_m\}$ lies on an affine hyperplane then I_A is a homogeneous ideal and $X_A = V(I_A) \subseteq \mathbb{P}^n$.