Affine toric variety

$\mathbb{C}^* = \mathbb{C} - \{0\}$

A toric variety is the closed image of a monomial map

$$\varphi_A : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}^m$$
$$(t_1, \dots, t_n) \longmapsto (t^{a_1}, \dots, t^{a_m}) \quad , \quad A = [a_1, \dots, a_m]$$

column vector

Two important groups : $\text{col space}_{\mathbb{Z}}(A) = \mathbb{Z}A$

subgroup of $\mathbb{Z}^m$

- $(\mathbb{C}^*)^n$

alg. torus

Recall : Every finitely generated free abelian group G is isomorphic to $\mathbb{Z}^m$ for some m. Say G has rank $= m$, $\text{rank}(G) = m$. $G \simeq \mathbb{Z}^m$

change of basis.

$(\mathbb{C}^*)^n = \{(t_1, \dots, t_n) \mid t_i \neq 0, t_i \in \mathbb{C}\} =$ group of invertible diagonal matrices of size $n \times n$

$= \{\text{diag}(t_1, \dots, t_n) \mid t_i \neq 0, t_i \in \mathbb{C}\}$

Associate $\mathbb{Z}^n$ to $(\mathbb{C}^*)^n$ in two ways

as algebraic groups = group + variety

1) $\mathbb{Z}^n \simeq \text{Hom}_{\mathbb{Z}}(\mathbb{C}^*, (\mathbb{C}^*)^n)$

each element of Hom is identified with $w = (w_1, \dots, w_n) \in \mathbb{Z}^n$ which gives a hom.

$$\mathbb{C}^* \longrightarrow (\mathbb{C}^*)^n$$
$$t \longmapsto \text{diag}(t^{w_1}, \dots, t^{w_n})$$

elements of $\text{Hom}_{\mathbb{Z}}(\mathbb{C}^*, (\mathbb{C}^*)^n)$ are called co-characters.

2) $\mathbb{Z}^n \cong \underbrace{\mathrm{Hom}_{\mathbb{Z}}}((\mathbb{C}^*)^n, \mathbb{C}^*)$

An element of $\mathrm{Hom}_{\mathbb{Z}}((\mathbb{C}^*)^n, \mathbb{C}^*)$ is identified $a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ via the <u>Laurent monomial</u> $x^a = x_1^{a_1} \cdots x_n^{a_n}$

$$(\mathbb{C}^*)^n \longrightarrow \mathbb{C}^*$$

$$\mathrm{diag}(t_1, \dots, t_n) \longmapsto t_1^{a_1} \cdots t_n^{a_n} = t^a$$

call elements of $\mathrm{Hom}_{\mathbb{Z}}((\mathbb{C}^*)^n, \mathbb{C}^*)$ characters (group characters of $(\mathbb{C}^*)^n$)

<u>Notation</u> A group isomorphic to $\mathbb{Z}^n$ is called a <u>lattice</u>.

N = lattice of cocharacters = $\mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n)$

M = lattice of characters = $\mathrm{Hom}((\mathbb{C}^*)^n, \mathbb{C}^*)$

Applying a character $a \in M$ to a cocharacter $w \in N$ we get a character of $\mathbb{C}^*$

$$N \otimes M \to \mathbb{Z}$$

$$w \otimes a \mapsto w \cdot a$$

$$\mathbb{C}^* \xrightarrow{w} (\mathbb{C}^*)^n \xrightarrow{a} \mathbb{C}^*$$

$$t \longmapsto \mathrm{diag}(t^{w_1}, \dots, t^{w_n}) \longmapsto (t^{w_1})^{a_1} \cdots (t^{w_n})^{a_n}$$

$$= t^{w_1 a_1 + \dots + w_n a_n} = t^{w \cdot a}$$

Def Coordinate ring of $(\mathbb{C}^*)^n = \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$

ring of Laurant Polynomials.

$= \mathbb{C}[M]$ = Group algebra of the lattice M

Aside coord ring of $\mathbb{C}^n$ is $\mathbb{C}[x_1, \dots, x_n]$

Aside we call $(\mathbb{C}^*)^n$ (or $(k^*)^n$) an algebraic torus

since $(\mathbb{C}^*)^2 \cong (\mathbb{R}_{>0} \times \underset{\text{circle}}{S^1})^2 = \underbrace{(\mathbb{R}_{>0} \times \mathbb{R}_{>0})}_{\text{contractible}} \times \underbrace{(S^1 \times S^1)}_{\text{topological torus}}$

For $A = [a_1, \dots, a_m]$ an integer matrix

Notation $(\mathbb{C}^*)^A = (\mathbb{C}^*)^{|A|} = (\mathbb{C}^*)^m$

$\mathbb{C}^A = \overline{(\mathbb{C}^*)^m} > \mathbb{C}^m$

Def (Affine toric variety) given $A \subseteq M$ (A finite subset), define a map

$$\varphi_A : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}^A = \mathbb{C}^m$$

$$(t_1, \dots, t_n) \mapsto (t^{a_1}, \dots, t^{a_m}) = (t^a \mid a \in A)$$

$$X_A = \overline{\varphi_A((\mathbb{C}^*)^n)}.$$

Note this is a composition of a group hom $(\mathbb{C}^*)^n \to (\mathbb{C}^*)^m$ and an inclusion $(\mathbb{C}^*)^m \hookrightarrow \mathbb{C}^m$.

$\varphi_A((\mathbb{C}^*)^n)$ is a subtorus of $(\mathbb{C}^*)^m$

If $\mathbb{Z}A$ is a proper subgroup of $M \simeq \mathbb{Z}^n$, then

φ_A has a (non-trivial) kernel $T := \ker(\varphi_A)$

as a map of groups

In Ex.2 you will show in this case there is a injective map

$$(\mathbb{C}^*)^n / T \longrightarrow \mathbb{C}^m$$

Further $\mathbb{Z}A$ = character lattice of $(\mathbb{C}^*)^n/T$.

Corollary 1 $\dim(X_A) = \dim\left((\mathbb{C}^*)^n/T\right) = \operatorname{rank} \mathbb{Z}A$

Proof sketch

dim of variety = dim of dense subset

$$\dim(X_A) = \dim\left(\overline{\varphi_A((\mathbb{C}^*)^n)}\right) = \dim\left(\varphi_A((\mathbb{C}^*)^n)\right)$$

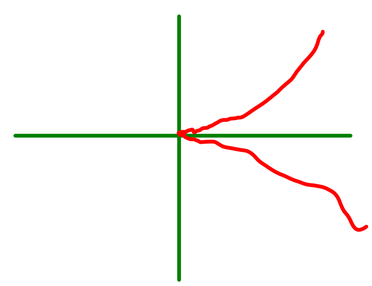
$$= \dim\left((\mathbb{C}^*)^n/T\right)$$

$$= \operatorname{rank}(\mathbb{Z}A)$$

Ex Take $n=1$, $A = \{2,3\} \subseteq \mathbb{Z}$

$$\varphi_A : \mathbb{C}^* \longrightarrow \mathbb{C}^2$$
$$t \longmapsto (t^2, t^3)$$

$$\overline{\varphi_A(\mathbb{C}^*)} = V(y^2 - x^3) \longleftrightarrow$$



Note $\mathbb{Z}A = \operatorname{Span}_{\mathbb{Z}}(2,3) = \operatorname{Span}(\gcd(2,3)) = \mathbb{Z}$

$\therefore$ φ_A is injective

Alt. $(x,y) = (t^2, t^3) \Rightarrow t = \frac{y}{x}$ injective from $\mathbb{C}^*$

Ex Take $d \geq 1$. The d^{th} Veronese map

$$\varphi_A : \mathbb{C}^n \longrightarrow \mathbb{C}^{\binom{d+n}{n}}$$

is given by taking $A =$ Set of all vectors $a \in \mathbb{N}^n$ s.t. $|a| = \deg(x^a) \leq d$

when $n=1$, $d=3$ $A = \{0, 1, 2, 3\}$

$$\varphi_A(t) = (1, t, t^2, t^3)$$

Toric Ideals

Let X_A be a toric variety, then

$$\underbrace{I_A}_{\text{toric ideal}} := I(X_A) = \text{all poly which vanish on all points in } X_A$$

we will show I_A is always generated by binomials.

Lemma I_A is a prime ideal

Proof $(\mathbb{C}^*)^n$ is an irreducible variety (defined by the ideal $\langle 0 \rangle$ in $\mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$)

$\therefore X_A = \overline{\varphi_A((\mathbb{C}^*)^n)}$ is irreducible since $(\mathbb{C}^*)^n$ ■

idea is that

$\underbrace{\varphi_A((\mathbb{C}^*)^n)}_{W}$ is irreducible, picking X_A as the smallest variety containing W

If $W \subseteq X_A = Z \cup Y$

Since W is image of irre. it would be in either Z or Y

$A = [a_1, \dots, a_m]$

Consider the pull back map induced by φ_A on coordinate rings

$$\varphi_A^*: \underbrace{\mathbb{C}[z_1, \dots, z_m]}_{\text{coord. ring of } \mathbb{C}^m} \longrightarrow \underbrace{\mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]}_{\text{of } (\mathbb{C}^*)^n}$$

$$z_i \longmapsto x^{a_i}$$

Note $R = \mathbb{C}[z_1, \dots, z_m]$

$$I(X_A) = \{ f \in R \mid f(z_1, \dots, z_m) = 0,\ z \in X_A \}$$

$$= \{ f \in R \mid f(x^{a_1}, \dots, x^{a_m}) = 0,\ x \in (\mathbb{C}^*)^n \}$$

$$= \ker(\varphi_A^*) .$$

as a map of ring

Let $u \in \mathbb{N}^m$, then

$$\varphi_A^*(z^u) = \varphi_A^*(z_1^{u_1} \cdots z_m^{u_m})$$

scalar, vector

$$= (x^{a_1})^{u_1} \cdots (x^{a_m})^{u_m} = x^{u_1 a_1 + \cdots + u_m a_m}$$

$$= x^{\underbrace{Au}_{\text{matrix vector product}}}$$

Au = vector of length n

Consider the binomials

$$\{ z^u - z^v \mid Au = Av \} \subseteq \ker(\varphi_A^*) = I_A$$

These binomials are in $\ker(\varphi_A^*)$ since

$$\varphi_A^*(z^u - z^v) = \underbrace{x^{Au} - x^{Av} = x^{Au} - x^{Au}}= 0$$

$$\therefore\ z^u - z^v \in \ker(\varphi_A^*)$$

Lemma Let X_A be a toric variety defined $A = [a_1, \dots, a_m]$

1) The binomials $\{ z^u - z^v \}$ are in I_A

2) Every binomial in I_A is in $\boxed{\{ z^u - z^v \mid Au = Av \}}$ (*)

3) $I_A = \langle z^u - z^v \mid Au = Av \rangle$, and in particular

$$I_A = \mathrm{Span}_{\mathbb{C}}(z^u - z^v \mid Au = Av)$$

Proof / 1) Proved already, 2 follows from 3).

Prove 3) Show $I_A = \mathrm{Span}_{\mathbb{C}}(z^u - z^v \mid Au = Av)$.

Fix a monomial order $<$ on $\mathbb{C}[z_1, \dots, z_m]$, let $f \in I_A$

Then can write

$$f = \underbrace{c_u z^u}_{= \mathrm{in}_<(f)} + \sum_{v<u} c_v z^v \quad , \; c_u \neq 0$$

Then $f \in I_A$; $\therefore f \in \ker(\varphi_A^*)$, so

$$0 = \varphi_A^*(f) = c_u x^{Au} + \sum_{v<u} c_v x^{Av}$$

as a poly in x

$\therefore$ Must exist some $v < u$ with $Av = Au$

otherwise $c_u x^{Au}$ is ~~not~~ canceled in $\varphi_A^*(f)$ and $\varphi_A^*(f) \neq 0$.

Pick some v with $Au = Av$

Set $\bar{f} = f - c_u(z^u - z^v)$

then $\varphi_A^*(\bar{f}) = \underbrace{\varphi_A^*(f)}_{=0 \text{ by assumption}} - c_u \underbrace{\varphi_A^*(z^u - z^v)}_{=0 \text{ since } Au = Av}$

then $\varphi_A^*(\bar{f}) = 0$, and $\bar{f} \in \ker(\varphi_A^*)$

Either $\mathrm{in}_<(f)$ is minimal in $\mathrm{in}_<(I_A)$, or it isn't

If it is minimal then $\bar{f} = 0$ since

$$\mathrm{in}_<(\bar{f}) < \mathrm{in}_<(f)$$

By induction we may suppose that if

$in_<(f)$ is not minimal in $in_<(I_A)$ then every poly in I_A, having only terms which are < less than $in_<(f)$ is a $\mathbb{C}$-linear combination of the desired binomials (*)

But $\bar{f} < f \Rightarrow \bar{f}$ is a $\mathbb{C}$-linear combo of binomials (*), But

$$f = \underbrace{\bar{f}}_{\text{is } \mathbb{C}\text{-linear combo of binomials (*) by Induction}} + c_u \underbrace{(z^u - z^v)}_{\text{a (*) binomial}}$$

$\therefore$ f is a $\mathbb{C}$-linear combo of binomials (*) ∎

Remark

$$I_A = \underbrace{\langle z^u - z^v \mid Au = Av \rangle}_{\text{infinite generating set, a priori}}$$

By Hilbert's basis theorem, some finite subset of these binomials must also generate I_A.

A: Markov Basis for col. space of A.

A <u>semi-group</u> is a set with an associative binary operation, but may not have an identity.

$$\mathbb{N}A = \text{sub semigroup of } M \text{ generated by the columns of } A$$
$$= \text{lin combo. of columns of } A \text{ with } \mathbb{N} \text{ coefficients.}$$

$\mathbb{C}[\mathbb{N}A]$ = Set of Laurent polynomials whose exponents are in $\mathbb{N}A$, i.e. all monomials are x^a for $a \in \mathbb{N}A$.

$\doteq$ Semigroup algebra on $\mathbb{N}A$

Proposition / The coordinate ring of X_A is $\mathbb{C}[\mathbb{N}A]$

Proof / The map

$$n_1a_1 + \cdots + n_m a_m \longmapsto (x^{a_1})^{n_1} \cdots (x^{a_m})^{n_m}$$

is a bijection between $\mathbb{N}A$ and the semi-group of monomials

$$\underbrace{(x^a \mid a \in A)}_{\text{generated by } A}$$

$\varphi_A^*(\mathbb{C}[z_1, \dots, z_m])$ is the subalgebra of $\mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ generate by the monomials $\{x^a \mid a \in A\}$, its kernel is I_A

$$\mathbb{C}[X_A] = \mathbb{C}[z_1, \dots, z_m]/I_A = \mathbb{C}[z_1, \dots, z_m]/\ker(\varphi_A^*)$$

$\therefore$ φ_A^* gives an isomorphism

$$\mathbb{C}[z_1, \dots, z_m]/\ker(\varphi_A^*) \longrightarrow \text{subalgebra of } \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}] \text{ generated by } \{x^a \mid a \in A\}$$

The conclusion follow by the bijection $\longleftrightarrow \mathbb{N}A$ ∎