Weil Divisors

Def] A weil divisor on a normal variety X is a finite formal sum

$$D = \sum_{i=1}^{s} a_i D_i$$

where D_i are distict irreducible divisors of X and $a_i \in \mathbb{Z}$.

Given a non-zero rational function $f \in (\mathbb{C}(X))^* = \left(\mathrm{frac}\left(\mathbb{C}[x_1,\dots,x_n]/I_X\right)\right)^*$ we can define $\mathrm{ord}_Y(f)$ for every irreducible divisor $Y \subseteq X$. This gives a weil divisor

$$\mathrm{div}(f) = \sum_Y \mathrm{ord}_Y(f)\, Y$$

This is a finite sum as there are at most finitely many divisors whose order is non-zero

Two weil divisors D_1, D_2 on X are linearly equivilent written $D_1 \sim D_2$ if $\exists$ $f \in \mathbb{C}(X)$ s.t.

$$\mathrm{div}(f) = D_1 - D_2$$

A weil Divisor D is called principal if $D \sim 0$ i.e. $D = \mathrm{div}(f)$ for some f

Def] (codim 1 chow Group / Divisor class group) Let X be normal.

Let $m = \dim(X)$. The dimension $(n-1)$ or codimension 1 Chow group $A_{m-1}(X) = A^1(X)$ is the group of Weil divisors on X modulo linear equivalence.
"$Cl(X)$"

Aside) In general Chow groups of dim $0, \dots, m$ can be defined via consider analogus finite formal sums

$\sum n_i [V_i]$ where V_i is an irreducible subvariety of X of dim $= 0 \le l \le m$

and $[V_i] =$ rational eq. class of V_i

E.g.) For $X = \mathbb{P}^n$, $A_l(\mathbb{P}^n) \cong \mathbb{Z}$ for all l

if V_1, V_2 are irr. subvarities of $\mathbb{P}^n$ of dim l

then $[V_1] \sim [V_2]$ iff $\deg(V_1) = \deg(V_2)$.

Weil Divisors on Toric Varieties

In this setup it helps to remember when

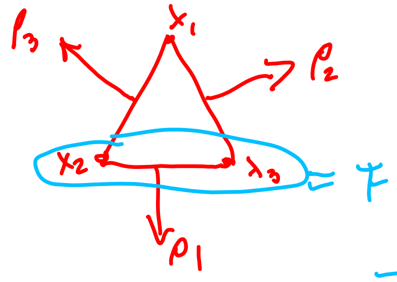
$\Sigma =$ outer normal fan of a polytope P

then the orbit/cone correspondence on X_A where $P = \mathrm{conv}(A)$

gives

$$X_A = \bigcup_{\substack{F \text{ a face} \\ \text{of } P}} \overline{\sigma(F)}$$

$$\overline{\sigma(F)} = \left\{ [y_0 : \dots : y_n] \in \mathbb{P}^n \;\middle|\; y_i = \begin{cases} 0 \text{ if } a_i \notin F \\ t^{a_i},\ t \in (\mathbb{C}^*)^r \text{ if } a_i \in F \end{cases} \right\}$$



For this face $\overline{\sigma(F)} = V(x_1) \cap X_A$

$\Updownarrow$

In the cox ring $\mathbb{C}[x_{\rho_1}, x_{\rho_2}, x_{\rho_3}]$ this corresponds to

taking $x_{\rho_1} = 0$ so $X_{\Sigma} \cap V(x_{\rho_1})$

In either case we get a Divisor of X_A / X_{Σ} from a facet/ray

Thm (Fan version of orbit/cone)

Let X_{Σ} is the normal toric variety of a fan Σ

Then $\exists$ a bijective correspondence

$$\{\text{cones in } \Sigma\} \longleftrightarrow \{(\mathbb{C}^*)^r \text{ orbits in } X_{\Sigma}\}$$

and $\sigma \longleftrightarrow \mathcal{O}(\sigma) \simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma^{\perp} \cap M, \mathbb{C}^*)$

Concretely if $U_\sigma = \overline{\mathcal{O}(\sigma)} \subseteq \mathbb{C}^n$ and τ is a face of σ

$\sigma = \mathrm{Cone}(a_1, \dots, a_n)$

then $\overline{\sigma(\tau)} = \left\{ (y_1, \dots, y_n) \mid y_i = \begin{cases} 0 & \text{if } a_i \in \tau \\ t^{a_i} & \text{if } a_i \notin \tau \end{cases} \right\}$

So in particular if ρ_i is a ray σ, $\rho_i = \mathbb{R}_{\geq 0}\{a_i\}$

then $\overline{\sigma(\rho_i)} = U_\sigma \cap V(x_i)$ is a divisor on U_σ.

In general for X_Σ a ray $\rho \in \Sigma$ corresponds to a divisor on each U_σ for which $\rho \in \sigma$, and hence on X_Σ. Denote this divisor as D_ρ.

For a cocharacter $m \in M$ we may also define a divisor

$$\mathrm{div}(t^m) = \sum_\rho \langle m, v_\rho \rangle D_\rho \quad \text{wher } \rho = \mathbb{R}_{\geq 0} v_\rho$$

for each $\rho \in \Sigma(1)$.

Note that t^m maps $(\mathbb{C}^*)^r \longrightarrow \mathbb{C}^*$

So we can think of this as a rational function on X_Σ which is non-vanishing on $(\mathbb{C}^*)^r$.

Since X_Σ is normal an D_ρ are irreducible then the order of vanishing is defined

and a theorem

$$\mathrm{Ord}_{D_\rho}(t^m) = \langle m, v_\rho \rangle \qquad \dim(X_\Sigma) = r$$

Fact | The chow group $A_{r-1}(X_\Sigma)$ is generated by D_ρ s.t. $\rho \in \Sigma(1)$. Further there is an exact sequence

$$0 \to M \xrightarrow{\alpha} \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} D_\rho \xrightarrow{\beta} A_{n-1}(X_\Sigma) \to 0$$

α is the map given by

$m \mapsto t^m \mapsto \sum \langle m, u_\rho \rangle D_\rho$

β is the map giving linear equivalence of divisors.

Example) Consider $X_\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$

$$\Sigma(1) = \{ \underset{\rho_1}{(1,0)}, \underset{\rho_2}{(-1,0)}, \underset{\rho_3}{(0,1)}, \underset{\rho_4}{(0,-1)} \} \qquad \rho_i = \mathbb{R}_{\geq 0} v_i$$

Then the exact sequence is

$$0 \to \underline{\underline{\mathbb{Z}^2}} \to \bigoplus_{i=1}^{4} \mathbb{Z}_{D_{\rho_i}} \xrightarrow{\beta} \underset{t_1^a t_2^b}{\mathbb{Z}^2} \to 0$$

where $\alpha(a, b) = aD_1 - aD_2 + bD_3 - bD_4$

$$\beta = (a_1 D_1 + \cdots + a_4 D_4) = (a_1 + a_2, a_3 + a_4)$$

we can also think of the coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$

as $\mathbb{C}[x_0, x_1] \otimes \mathbb{C}[y_0, y_1] = \mathbb{C}[x_0, x_1, y_0, y_1]$

where $\deg(x_i) = (1,0)$, $\deg(y_i) = (0,1)$

Note $x_0 \leftrightarrow D_1$, $x_1 \leftrightarrow D_2$, $y_0 \leftrightarrow D_3$, $y_1 \leftrightarrow D_4$

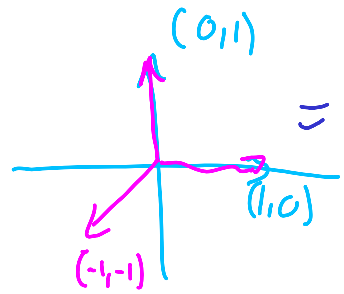
So we can read the map β as saying

$$\deg\left(x_0^{a_1}x_1^{a_2}y_0^{a_3}y_1^{a_4}\right) = \left(a_1+a_2, a_3+a_4\right)$$

tot. deg. in Y-var.

tot. deg in X-var

Def (Simplicial fan) A cone $\sigma = \mathrm{Cone}(v_1,\dots,v_r)$ is called simplicial if it's minimal generators are linearly independent over $\mathbb{R}$. A fan Σ is called simplicial if every cone $\sigma \in \Sigma$ is simplicial.

E.g. $\mathbb{P}^2$ has the fan  $= \Sigma$

maximal cones of Σ are generated by any two rays is lin ind over $\mathbb{R}$

Σ - Simplicial fan

$$R = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$$

So we grade this ring as follows:

- Note the Divisor D_ρ corresponds to x_ρ, or e.q. to $V(x_\rho) \cap X_\Sigma \subseteq \mathbb{C}^{\Sigma(1)}$

So we grade R by Pairing

$$X^D := \prod_{\rho \in \Sigma(1)} x_\rho^{a_\rho} \iff D = \sum a_\rho D_\rho$$

So $\deg(x^D) = \beta(D)$

where β is from the exact seq.

$$M \longrightarrow \bigoplus_\rho \mathbb{Z} D_\rho \xrightarrow{\beta} A_{n-1}(X_\Sigma) \longrightarrow 0$$

So $\beta(D)$ is represented as an integer vector

$$B = \left\langle \prod_{\rho \in \sigma} x_\rho \;\middle|\; \sigma \in \Sigma \right\rangle \subseteq R$$

<u>Thm</u> $X_\Sigma \cong \left(\mathbb{C}^{\Sigma(1)} - V(B)\right) / G$ where

$G = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X_\Sigma), \mathbb{C}^*)$ with $n = \dim(X_\Sigma)$
and Σ a simplicial fan.

$$x, y \in \left(\mathbb{C}^{\Sigma(1)} - V(B)\right) / G$$

$x \sim y$ iff $x \in y \cdot G$

<u>Toric ideal-variety correspondence</u>

$$\pi : \left(\mathbb{C}^{\Sigma(1)} - V(B)\right)/G \xrightarrow{\sim} X_\Sigma$$

given $p \in X_\Sigma$ we say $x \in \pi^{-1}(p)$ gives <u>homogeneous coordinates</u> for p and we know

$$\pi^{-1}(p) = G \cdot x$$

If we have $f \in R$ which is homogeneous w.r.t to the grading by β, i.e. each monomial x^α of f has $\deg(x^\alpha) = D \in \mathrm{Cl}(X_\Sigma)$

$$f(g \cdot x) = (t_g)^{\deg(f)} f(x)$$

$\therefore$ the equation $f(x) = 0$ is well defined on X_Σ

<u>Prop</u> Let R be the cox ring of the simplicial toric variety X_Σ. Then

- If $I \subseteq R$ is a homogeneous ideal w.r.t. the grading by $\mathrm{Cl}(X_\Sigma)$ then

$$V(I) = \{ \pi(x) \in X_\Sigma \mid f(x) = 0 \ \forall f \in I \}$$

is a closed subvariety of X_Σ.

- all closed subvarieties arise in this way in $\mathbb{C}$

$\mathbb{C}^n$ $\{$closed subvarieties of $\mathbb{C}^n\} \overset{1-1}{\longleftrightarrow} \{$radical ideals $I \subseteq \mathbb{C}[x_1, \dots, x_n]\}$

$\mathbb{P}^n$ $\{$closed subvarieties of $\mathbb{P}^n\} \longleftrightarrow \{$radical homogeneous ideals $I \subseteq \mathbb{C}[x_0, \dots, x_n]\}$

Def we say an ideal $I \subseteq R$ (← cox ring of X_Σ) is B-saturated

if $I : B^\infty = I$

Thm Let X_Σ be a smooth toric variety then we have the following bi-jective correspondence

$\{$closed subvarieties of $X_\Sigma\} \longleftrightarrow \{$radical B-saturated homogenous ideals (w.r.t grading by $A_{r-1}(X_\Sigma)$) $I \subseteq R\}$

where $R =$ cox ring of X_Σ, $r = \dim(X_\Sigma)$.

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keep in mind

$\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\}) / \mathbb{C}^*$

$\lambda \cdot (p_0, \dots, p_n) = (p_0, \dots, p_n)$ in

Let G be a group acting on a variety X.

Assume that for every $g \in G$ the map

$\phi_g(x) = g\cdot x$ defines a morphism $\phi_g : X \to X$.

when $X = \mathrm{Spec}(R)$ is affine, i.e. $X \subseteq \mathbb{C}^n$,

$\phi_g : X \to X$ comes from $\phi_g^* : R \to R$

we define the induced action of G on R by

$$g\cdot f = \phi_{g^{-1}}^*(f) \quad \text{for } f \in R$$

i.e. $$(g\cdot f)(x) = f(g^{-1}\cdot x) \quad \forall x \in X.$$

Two objects

- Set of G-orbits $X/G = \{G\cdot x \mid x \in X\}$
- The ring of invariants $R^G = \{f \in R \mid g\cdot f = f \ \forall g \in G\}$

Note $R^G \subseteq R$, if $f \in R^G$ then

$$\bar{f}(G\cdot x) = f(x)$$

gives a well defined function $\bar{f} : X/G \to \mathbb{C}$

- would like to identify X/G with $\mathrm{Spec}(R^G)$

Issues: R^G is <u>not</u> always finitely generated

but it is in nice case, e.g. tori $(\mathbb{C}^*)^r$

we say a subgroup $G \subseteq Gl_n(\mathbb{C})$ is an affine algebraic group if, it is also a subvariety (not nessarily closed) of $Gl_n(\mathbb{C})$

Eg $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $(\mathbb{C}^*)^n$, finite groups

An affine alg. group acts algebraically on a variety $X \subseteq \mathbb{C}^n$ if the G-action $g \cdot x$ defines a morphism of varieties

$$G \times X \to X$$

A group is called reductive if it's maximal connected solvable subgroup is a torus

e.g. | finite groups, tori, semi-simple groups, $SL_n(\mathbb{C})$

Prop | Let G be a reductive group acting algebraically on an affine variety $X = Spec(R)$. Then

(a) R^G is finitely generated (as a $\mathbb{C}$-algebra)

(b) For the morphism $\pi : X \to Spec(R^G)$ the following are equivalent

(i) All G-orbits are closed in X

(ii) Given $x, y \in X$ we have

$\pi(x) = \pi(y)$ iff x and y are in the same G-orbit

(iii) π induces a bijection

$$\left\{ G\text{-orbits in } X \right\} \longleftrightarrow \operatorname{Spec}(R^G)$$

$$\|$$

$$X/G$$