Normal Varieties

Let R be an integral domain with field of fractions $K = \text{frac}(R)$. Then R is integrally closed if every element $a \in K$ which is integral over R is in R

A element $a \in K$ is integral over R if it is the root of a monic polynomial in $R[x]$

Recall a variety V is irreducible iff $V = Z \cup W$ $\Rightarrow V = Z$ or $V = W$ (Z, W are varieties)

There is a 1-1 correspondence between prime ideals and irreducible varieties

If $V = V(I) \subseteq \mathbb{C}^n$ is an irr. var. defined by a prime ideal I then its coordinate ring $\mathbb{C}[x_1, \dots, x_n]/I$ is an integral domain

Def/ A irreducible variety $V = V(I) \subseteq \mathbb{C}^n$ is normal if $\mathbb{C}[x_1, \dots, x_n]/I$ is integrally closed.

Toric varieties defined by the closure of a monomial map need not be normal.

Ex/ Consider $A = \{2, 3\}$ defining the monomial map

$$\varphi_A(s,t) \longmapsto (s^2, t^3)$$

The associated toric ideal is $I_A = \langle x^3 - y^2 \rangle \subseteq \mathbb{C}[x,y]$

and the coordinate ring is

$$R = \mathbb{C}[x,y]/\langle x^3 - y^2 \rangle$$

Consider the poly $f(w) \in R[w]$ given by

$$f(w) = w^2 - x \quad \text{and the element } \alpha = \frac{y}{x} \in \operatorname{frac}(R)$$

Note in $\operatorname{frac}(R)$ $\quad y^2 = x^3$

Hence

$$f\left(\frac{y}{x}\right) = \left(\frac{y}{x}\right)^2 - x = \frac{x^3}{x^2} - x = 0$$

$\therefore$ $\frac{y}{x}$ is a root of a poly in $R[w]$, but $\frac{y}{x} \notin R$

$\therefore$ R is <u>not</u> integrally closed and

$X_A = V(I_A)$ is <u>not</u> normal.

Thm/ Let $\sigma \subset N_{\mathbb{R}} \simeq \mathbb{R}^n$ be a pointed rational polyhedral cone. Let $V_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma])$, $S_\sigma = \sigma^\vee \cap M$. Then V_σ is a normal toric variety of dimension n.

Proof sketch/

We know from before S_σ is finitely generated say by $m_1, \ldots, m_r$. Then if $A = [m_1, \ldots, m_r] \in \mathbb{Z}^{n \times n}$ we know $V_\sigma \simeq X_A$ and that $\dim(X_A) = n$ whenever $\mathbb{Z}A = \mathbb{Z}^n$ (and this always holds for S_σ)

Now show V_σ is normal, i.e. show

$\mathbb{C}[x_1, \ldots, x_n]/I_A \simeq \mathbb{C}[S_\sigma]$ is integrally closed.

Aside | $H_u^+ = \{ v \in \mathbb{R}^n \mid \langle u, v \rangle \geq 0 \}$

Recall $\sigma^\vee = \{ u \in \mathbb{R}^n \mid \langle w, u \rangle \geq 0 \ \forall w \in \sigma \}$

$\therefore$ $\sigma \subseteq H_u^+$ iff $u \in \sigma^\vee$

Thm | If $\sigma = H_{u_1}^+ \cap \cdots \cap H_{u_s}^+ \subseteq \mathbb{R}^n$ is a convex pointed polyhedral cone then $\sigma^\vee = \mathbb{R}_{\geq 0} u_1 + \cdots + \mathbb{R}_{\geq 0} u_s$ and $(\sigma^\vee)^\vee = \sigma$, and hence, if $\sigma = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_r$

$\Rightarrow \sigma^\vee = H_{v_1}^+ \cap \cdots \cap H_{v_r}^+$.

Show $\mathbb{C}[S_\sigma]$ is integrally closed.

Using $S_{\tau \cap \sigma} = S_\tau + S_\sigma$

If we write $\sigma^\vee = H_{v_1}^+ \cap \cdots \cap H_{v_r}^+$ where $\sigma = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_r$ and set $\tau_i = \mathbb{R}_{\geq 0} v_i$

then

$$\mathbb{C}[S_\sigma] = \mathbb{C}[\sigma^\vee \cap M] = \bigcap_{i=1}^{r} \mathbb{C}[\tau_i^\vee \cap M] = \bigcap_{i=1}^{r} \mathbb{C}[S_{\tau_i}]$$

Lemma | If τ is a ray in $\mathbb{R}^n$ defining a pointed cone $\sigma = \tau + \rho_1 + \cdots + \rho_s$ then $\mathbb{C}[S_\tau] \cong \mathbb{C}[x_1, x_2^\pm, \ldots, x_n^\pm]$ where the 1st coordinate is fixed by taking the first element of a $\mathbb{Z}$-basis of N as τ

geometrically this is saying that V_τ has only one zero coordinate when embedded in $\mathbb{C}^n$

Now $\mathbb{C}[S_{\tau_i}] \cong \mathbb{C}[x_1, x_2^\pm, \ldots, x_n^\pm] \simeq (\mathbb{C}[x_2^\pm, \ldots, x_n^\pm])[x_1]$

Is a UFD.

Since The Laurent Poly ring is a UFD and $R[x]$ is a UFD if R is.

Now let $R = \mathbb{C}[S_{\tau}]$ which is a UFD, we wish to Show that R is integrally closed

Let $f \in R[T]$ be a monic poly, $\alpha \in \mathrm{frac}(R)$ be a root of f.

then $\alpha = \frac{a}{b}$, $a, b \in R$. By Unique fact. we may assume that no irreducible element of R divides both a and b.

If $f(T) = T^n + c_{n-1}T^{n-1} + \cdots + c_0$ then

$f(\alpha) = 0$ and $b^n f(\alpha) = 0$

so

$$0 = b^n f(\alpha) = a^n + c_{n-1} b a^{n-1} + \cdots + c_0 b^n.$$

$$\therefore \quad -a^n = \underbrace{c_{n-1} b a^{n-1} + \cdots + c_0 b^n}_{\text{is divisable by } b}$$

$\therefore$ a^n is divisable by b

$\therefore$ Since no irreducible divides both a and b and b divides $a^n \Rightarrow b$ is a unit in R $\therefore \frac{a}{b} \in R$.

$\therefore$ R is integrally closed.

Thm Let V be an affine toric variety. Then V is isomorphic to $V_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$ for a pointed rational polyhedral cone iff V is normal.

With some care, can show that X_Σ is a normal toric variety since V_σ is a normal toric variety $\forall \sigma \in \Sigma$.

Notation Given a fan $\Sigma \subseteq N_{\mathbb{R}}$ let $\Sigma(1)$ denote the set of rays of Σ (i.e. all 1-dim cones in Σ).

Global coordinate ring for the toric variety X_Σ.

Consider $\mathbb{P}^n = X_\Sigma$, where Σ = fan which defines $\mathbb{P}^n$ as a toric variety

where Σ is the fan with rays

$$\Sigma(1) = \{e_1, \dots, e_n, -(e_1 + \dots + e_n)\}$$

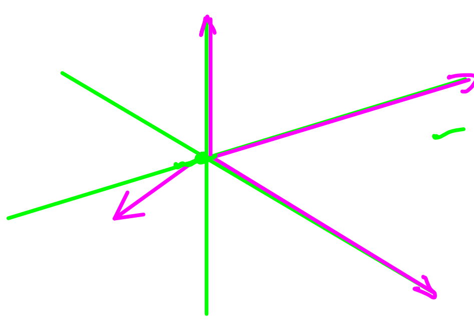
standard basis vectors for $\mathbb{Z}^n$ ($\mathbb{R}^n$)

and maximal cones

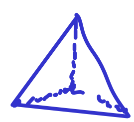
$$\Sigma(n-1) = \mathrm{Cone}(w_1, \dots, w_n) = \mathbb{R}_{\geq 0} w_1 + \dots + \mathbb{R}_{\geq 0} w_n$$

for each unique subset $\{w_1, \dots, w_n\}$ of $\Sigma(1)$.

E.g. $\mathbb{P}^3$, $\Sigma(1) = \{(1,0,0), (0,1,0), (0,0,1), -(1,1,1)\}$



outer normal fan of a triangular pyramid



$\mathbb{P}^3 \cong X_A$ for

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$(t_1, t_2, t_3, t_4) \longmapsto [t_1 t_4 : t_1 t_3 : t_1 t_2 : t_1]$$

The usual homogeneous coordinate ring for $[t_4 : t_3 : t_2 : 1]$

$\mathbb{P}^n$ is $\mathbb{C}[x_0, \dots, x_n]$ where two radical ideals define the same projective variety X iff

$$I : \langle x_0, \dots, x_n \rangle^\infty = J : \langle x_0, \dots, x_n \rangle^\infty$$

Aside $V(I : J^\infty) = \overline{V(I) - V(J)}$

we call $B = \langle x_0, \dots, x_n \rangle$ the irrelevant ideal

since points in $\mathbb{P}^n$ are represented by points in $(\mathbb{C}^{n+1} - V(B))/\sim$

$$V(I) \cup V(B) = V(I \cdot B) = V(I) \quad \forall I$$

E.g. in $\mathbb{P}^2$

$I = \langle x_0 - x_1, x_2^2 - 4x_0 \rangle$ defines

the variety

$$V(I) = \{p_1, p_2\} = \{[1:1:2], [1:1:-2]\}$$

$$J = \langle x_0 x_2 - x_1 x_2, x_0 x_1 - x_1^2, x_0^2 - x_1^2, x_2^3 - 4x_1 x_2, x_1 x_2^2 - 4x_1^2 \rangle$$
$$= I \cdot \langle x_0, x_1, x_2 \rangle$$

$V(J) = \{p_1, p_2\}$. and $I = J : B^\infty$

Note that since for Σ the fan of $\mathbb{P}^n$, $|\Sigma(1)| = n+1$

we can write $\mathbb{C}[x_0, \dots, x_n]$ as $\mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$

Note that we can also define B as follows

set $\sigma(1) = \{\rho \in \Sigma(1) \mid \rho \text{ is a face of } \sigma\}$

$$B = \left\langle \prod_{\rho \notin \sigma(1)} x_\rho \;\middle|\; \sigma \in \Sigma \right\rangle$$

Lets check this works for $\mathbb{P}^3$ set $\hat{e} = -(e_1 + e_2 + e_3)$

then the max cones are $\sigma_1 = \text{Cone}(e_1, e_2, e_3)$, $\sigma_2 = \text{Cone}(\hat{e}, e_1, e_2)$

$\sigma_3 = \text{Cone}(\hat{e}, e_2, e_3)$, $\sigma_4 = \text{Cone}(\hat{e}, e_1, e_3)$

label $\rho_0 = \mathbb{R}_{\geq 0}\hat{e}$, $\rho_i = \mathbb{R}_{\geq 0} e_i$ $i = 1, \dots, 3$

$\rho_0 \notin \sigma_1(1)$, $\rho_3 \notin \sigma_2(1)$, $\rho_1 \notin \sigma_3(1)$, $\rho_2 \notin \sigma_4(1)$

$$B = \langle x_{\rho_0}, x_{\rho_3}, x_{\rho_1}, x_{\rho_2} \rangle \xleftrightarrow{x_i = x_{\rho_i}} \langle x_0, x_1, x_2, x_3 \rangle$$

Fixing a fan Σ

Now define $S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$

$$B = \left\langle \prod_{\rho \notin \sigma(1)} x_\rho \;\middle|\; \sigma \in \Sigma \right\rangle \subseteq S$$

Set $\mathbb{C}^{\Sigma(1)} = \text{Spec}(S) \cong \mathbb{C}^{|\Sigma(1)|}$

Then the coordinates for points in

X_Σ will be choosen from

$$\mathbb{C}^{\Sigma(1)} - V(B)$$

with some equivalence relation

i.e. like in $\mathbb{P}_n$, $(\mathbb{C}^{n+1} - \{0\})/\sim$

where $p_1 = p_2 \in \mathbb{P}^n$ iff $p_1 = \lambda p_2 \in \mathbb{C}^{n+1} - \{0\}$.

Thm 1 (Cox) $X_\Sigma \cong (\mathbb{C}^{\Sigma(1)} - V(B))/G$ ← Quotient by a group action where

$G = \mathrm{Hom}_{\mathbb{Z}}(\underbrace{A^1(X_\Sigma)}_{\text{Codim 1 chow group of } X_\Sigma \text{ / Divisor class group of } X_\Sigma}, \mathbb{C}^*)$ where Σ is a simplicial fan.

i.e. back to $\mathbb{P}^n$

$$\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^* \qquad - G = \mathbb{C}^*$$

$p_1 \sim p_2$ iff p_1 and p_2 are in the same $\mathbb{C}^*$ orbit

i.e. $p_2 \in G \cdot p_1$

$$G p_1 = (\mathbb{C}^*) \cdot p_1 = \{ g \cdot p_1 \mid g \in \mathbb{C}^* \}$$

So the Quotient $\left(\mathbb{C}^{\mathcal{E}(1)} - V(\mathcal{B})\right)/G$ means

points

$$P_1, P_2 \in \mathbb{C}^{\mathcal{E}(1)} - V(\mathcal{B}) \text{ are identified}$$

if $P_2 \in G \cdot P_1$

The Divisor class group

Let $X = V(I) \subseteq \mathbb{C}^n$ be an irreducible variety defined by a prime ideal I.

An irreducible subvariety $Y \subseteq X$ is called an irreducible divisor if $\operatorname{codim}(Y, X) = 1$.

Def: let $Y \subseteq X = V(I) \subseteq \mathbb{C}^n$ be an irreducible divisor and set

$$\mathcal{O}_{X,Y} := \left\{ f \in \underbrace{\operatorname{frac}\left(\mathbb{C}[x_1,\dots,x_n]/I\right)}_{:= \mathbb{C}(X)} \;\middle|\; \begin{array}{l} f \text{ is defined on a} \\ \text{non-empty Zariski open set} \\ \text{of } Y \end{array} \right\}$$

Note: Every $f \neq 0 \in \mathbb{C}(X)$ is defined on some non-empty Zariski open $U \subseteq X$

(in particular $f = \frac{p}{q}$, so it is defined on $X - V(I + \langle q \rangle)$)

Then $f \in \mathcal{O}_{X,Y}$ when we can find such a U where $U \cap Y \neq \emptyset$

A ring is called a local ring if it has a unique maximal ideal.

Lemma $\mathcal{O}_{X,Y}$ is a local ring with maximal ideal consisting of $f \in \mathcal{O}_{X,Y}$ s.t. $f(p)=0 \ \forall p \in Y$.

Ex Let $Y = V(x) \subseteq \mathbb{C}^2$ (the y-axis)

$$\mathcal{O}_{\mathbb{C}^2,Y} = \left\{ \frac{p(x,y)}{q(x,y)} \;\middle|\; p,q \in \mathbb{C}[x,y],\ q(0,y) \neq 0 \right\}$$

One can show:

- Every $f \in \mathbb{C}(x,y)$ is $f = x^m g$, $m \in \mathbb{Z}$, $g \in \mathcal{O}_{\mathbb{C}^2,Y}$ where g is a unit
- every non-zero ideal in $\mathcal{O}_{\mathbb{C}^2,Y}$ is of the form $\langle x^m \rangle$ for some $m \geq 0$

This says given $f \in \mathbb{C}(x,y)$

$$\Rightarrow f = x^m g, \quad m \in \mathbb{Z},\ g \text{ a unit in } \mathcal{O}_{\mathbb{C}^2,Y}$$

we call m the <u>order of vanishing</u> of f on $Y = V(x) \subseteq \mathbb{C}^2$

This can be generalized:

Def (DVR) Let R be an integral domain with $K = \mathrm{frac}(R)$, set $K^* = K - \{0\}$. Then R is a <u>discrete valuation ring</u> if $\exists$ a surjective function

$$\mathrm{Ord}_R : K^* \longrightarrow \mathbb{Z}$$

s.t. for every $a, b \in K^*$ we have:

- $ord_R(a \cdot b) = ord_R(a) + ord(b)$

- $ord(a+b) \geq \min(ord_R(a), ord_R(b))$ whenever $a+b \neq 0$

- $R = \{a \in K^* \mid ord_R(a) \geq 0\} \cup \{0\}$

We say ord_R is a <u>valuation</u> of K and R is its <u>Valuation Ring</u>.