Ideals in Multivariant poly Rings

we will work in poly ring $R = k[x_1, .., x_n]$ for some field k. ussullay $k = \mathbb{C}, \mathbb{R}$ (theorems)
(for comp. $k = \mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$

R is an infinite dimensional v. space with distinshed basis of monomials

$$x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \longleftrightarrow (a_1, \dots, a_n) \in \mathbb{N}^n$$

If $f = \sum_a c_a x^a$, $\deg(f) = \max\{ |a| = a_1 + \cdots + a_n \mid c_a \neq 0 \}$

An (affine) algebebaic variety $X \subseteq k^n$ is defined by $f_1, .., f_r$

$$X = V(f_1, .., f_r) = \{ x \in k^n \mid f_1(x) = f_2(x) = \cdots = f_r(x) = 0 \}$$

If $I = \langle f_1, .., f_r \rangle = \langle g_1, .., g_s \rangle \subseteq k[x_1, .., x_n]$ is an ideal

then $V(f_1, .., f_r) = V(g_1, .., g_s)$

$\therefore$ we write $V(I) = V(f_1, .., f_r)$

Operations on poly. Ideals

Given ideals I, J in $R = k[x_1, .., x_n]$

i.e. sum of ideals $\longleftrightarrow$ intersection of varieties

- (sum) $I + J = \{ f + g \mid f \in I, g \in J \} \longleftrightarrow V(I + J) = V(I) \cap V(J)$
- (ideal intersection)

  $I \cap J = \{ f \mid f \in I, \text{ and } f \in J \} \longleftrightarrow$ union of varieties $V(I \cap J) = V(I) \cup V(J)$.
- Ideal quotient

  $I : J = \{ f \in R \mid f \cdot J \subseteq I \} \longleftrightarrow$

• Ideal Saturation $I : J^{\infty} = \{ f \in k \mid \exists n \in \mathbb{N} \text{ s.t. } f \cdot J^n \subseteq I \}$

$I : J^{\infty} = I : J^N$ for some N

$\mathbb{V}(I : J^{\infty}) = \overline{V(I) - V(J)}$ ← Zariski closure.

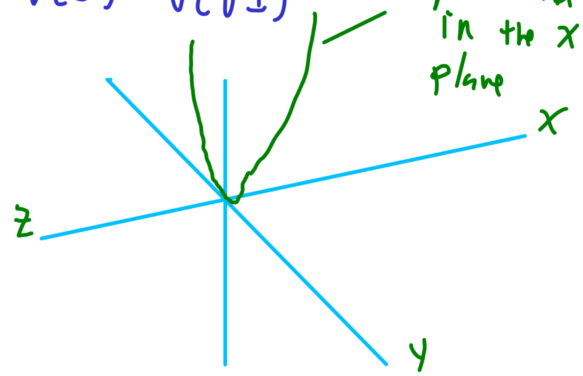
over an alg. closed field.

$\sqrt{I} = \{ r \in R \mid r^n \in I \text{ for some } n \in \mathbb{N} \}$

Check $I \subseteq \sqrt{I}$ and $V(I) = V(\sqrt{I})$

E.g. $\mathbb{V}(x^2, y^2 - z) = \mathbb{V}(x, y^2 - z)$

Parabola in the $x = 0$ plane



Types of ideals

Property	Def	R/I
I is maximal	no proper ideal contains I	is a field
I is Prime	$fg \in I \Rightarrow f \in I$ or $g \in I$	is an integral domain
I is radical	$\sqrt{I} = I$	has no nilpotent elements
I is primary	$\sqrt{I}$ is Prime	all zero divisors are nilpotent

Maximal ⇒ Prime ⇒ Primary

$\Downarrow$

radical

Monomial Orders and Gröbner basis

↓ order on set of monomials $x^a \in k[x_1, \dots, x_n]$

$\Updownarrow$

order on set of vectors $a \in \mathbb{N}^n$.

Def (Monomial Order) Consider the following total ordering $<$ on $\mathbb{N}^n$ (write $a \leq b$ if $a<b$ or $a=b$)

The order $<$ is a monomial order if $\forall a,b,c \in \mathbb{N}^n$

- $(0,\dots,0) \leq a$, i.e. $1 \leq x^a$
- $a < b \Rightarrow a+c < b+c$, i.e. $x^a < x^b \Rightarrow x^a \cdot x^c < x^b \cdot x^c$.

Ex Lex Order $a=(a_1,\dots,a_n)$, $b=(b_1,\dots,b_n)$

$x^a < x^b$ if $b_m - a_m > 0$ $\quad$ $m=$ last non zero entry in b

Degree/Graded lex order (Deglex or Glex)

$x^a < x^b$ if

- $|a| < |b|$, i.e. $\deg(x^a) < \deg(x^b)$
- if $|a| = |b|$ and $b_m - a_m > 0$

Degree/Graded Reverse lex order (Grevlex)

- $\deg(x^a) < \deg(x^b)$
- if $\deg(x^a) = \deg(x^b)$ and $b_m - a_m < 0$

Given a monomial order $<$ on $R = k[x_1,\dots,x_n]$ and $f \in R$

$in_<(f) = LM_<(f) =$ largest monomial in f w.r.t $<$

$f = 7x^2 + 3xz^2 + 2y^2$

$in_{lex}(f) = x^2$, $in_{Deglex}(f) = xz^2$, $in_{Grevlex}(f) = y^3$

$LC_<(f) =$ coefficient of the lead monomial

$LT_<(f) = LC_<(f) \cdot LM(f)$.

Def (Initial / Lead term ideal)

For an ideal I, the initial ideal or lead term ideal of I w.r.t $<$

$$in_<(I) = LT_<(I) = \langle LT_<(f) \mid f \in I - \{0\} \rangle$$

Fact (Dickson's lemma): we can always find a finite generating set for $LT_<(I)$.

Proposition / Definition Every ideal I has a finite set G s.t

$$LT_<(I) = \langle LT_<(f) \mid f \in G \rangle$$

The set G is called a Gröbner basis for I w.r.t. $<$.

Further $I = \langle g_1, \dots, g_r \rangle$ where $G = \{g_1, \dots, g_r\}$.

Cor (Hilbert's Basis Theorem) Every ideal I in $k[x_1, \dots, x_n]$ is finitely generated.

Def Fix $I, <$. A Gröbner basis G for I is reduced if

- $LC(g) = 1 \quad \forall g \in G$
- for $g \neq h$, $g, h \in G$, no monomial in g is a multiple of $LM_<(h)$.

Thm/ Every ideal I has a <u>unique</u> reduced Gröbner basis w.r.t. to $<$.

Upshot/ There is an algorithm to compute GB (Buchberger's Alg.)

This allows to automatically test equality of ideals.

The Zariski Topology

Idea: Put a topology on k^n where the closed sets are algebraic varieties $V(I)$.

Def/ (Irreducible variety) $V(I)$ is <u>irreducible</u> iff for all ideals J, J' in $k[x_1, \dots, x_n]$ we have

$$V(I) = V(J) \cup V(J') \Rightarrow \begin{array}{c} V(I) = V(J) \\ \text{or} \\ V(I) = V(J') \end{array} .$$

Ex/ $I = \langle xy \rangle = \langle x \rangle \cap \langle y \rangle$

$V(x) \subseteq V(I)$

$\langle x \rangle \supseteq I$

$$V(I) = V(x) \cup V(y)$$

These components are irreducible.

Def/ Given a subset $W \subseteq k^n$

$$I(W) = \{ f \in k[x_1, \dots, x_n] \mid f(p) = 0 \ \forall p \in W \}$$

Note:
- W is a variety iff $W = V(I(W))$
- If $V \subseteq W$ iff $I(W) \subseteq I(V)$
- $I(W)$ is a radical ideal.

Prop/ A variety $W \subseteq k^n$ is irreducible iff $I(W)$ is prime

Fact/ (Zariski topology) k^n is a topological space under the Zariski topology

- closed sets are all varieties $V(I)$ for some $I \subseteq k[x_1,\dots,x_n]$
- open sets are complements of closed sets, i.e

$$k^n - V(I) \text{ for some } I.$$

Zariski closure/ Given a set $U \subseteq k^n$ its Zariski closure is the smallest variety $V(I)$ s.t.

$$U \subseteq V(I).$$

write $\overline{U} = V(I)$.

Thm (Hilbert's Nullstelensatz) with k alg. closed Given an ideal J in $k[x_1,\dots,x_n]$

$$I(V(J)) = \sqrt{J}.$$

E.g.) $I(V(x^2,y)) = \langle x,y\rangle = \sqrt{\langle x^2,y\rangle}$

any commutative ring with Identity.

$k[x_1,\dots,x_n]$

Def/ The Spectrum of the ring R is the set of proper prime ideals

$$\mathrm{Spec}(R) := \{ p \subsetneq R \mid p \text{ is a prime ideal} \}$$

This is a topological space with the Zariski topology where the closed sets are the varieties

$$V(I) = \{ p \in \mathrm{Spec}(R) \mid I \subseteq p \}$$

Consider $\mathrm{Spec}(k[x_1, \ldots, x_n])$, this contains all

points $(p_1, \ldots, p_n) \in k^n$ represented by the

maximal ideal $\langle x_1 - p_1, \ldots, x_n - p_n \rangle$

However $\mathrm{Spec}(k[x_1, \ldots, x_n])$ also contains a "point" corresponding to each irreducible subvariety of k^n.

Fact: The Zariski topology on k^n is induced by the Zariski topology on $\mathrm{Spec}(k[x_1, \ldots, x_n])$

Def (Coordinate Ring of a variety) Given a subvariety $W \subseteq k^n$ define the coordinate ring of W

$$k[W] := k[x_1, \ldots, x_n] / I(W)$$

This induces a bijection

$$\{\text{prime ideals in } k[W]\} \longleftrightarrow \{\text{Prime ideals in } k[x_1, \ldots, x_n] \text{ which contain } J = I(W)\}$$

$\{\text{prime ideals in } k[W]\} \updownarrow \mathrm{Spec}(k[W])$

$\{\text{Prime ideals which contain } J\} \updownarrow$ all the irreducible varieties $V(p) \subseteq V(J) = W$, $p \supseteq J$

in particular each point $p = (p_1, \ldots, p_n) \in W$ corresponds to max. ideal of p, $\langle x_1 - p_1, \ldots, x_n - p_n \rangle$

Mapping, Projection, and Elimination

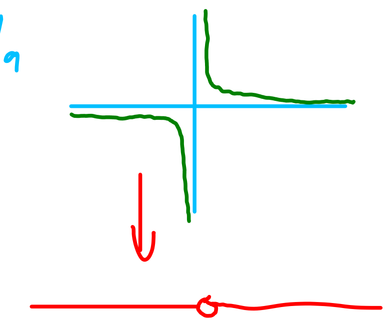
Consider the projection, $m < n$

$$\pi : k^n \longrightarrow k^m$$

$$(p_1, \dots, p_m, p_{m+1}, \dots, p_n) \longmapsto (p_1, \dots, p_m)$$

If V is a variety in k^n, $\pi(V)$ need not be.

Ex: $n = 2$, $m = 1$, $V = V(xy - 1)$ ← hyperbola



$\pi : k^2 \to k$ $\quad\quad \pi(V) = k - \{0\}$

↑ this is not a variety

$$\overline{\pi(V)} = k$$

Hence when looking at the image of a map, we take its Zariski closure to get closed image $\overline{\pi(V)}$ ≃ Zariski closure of $\pi(V)$.

Thm: Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal, $V = V(I) \subseteq k^n$, k alg. closed. $\pi : k^n \to k^m$ the projection with $m < n$, then

$$\overline{\pi(V)} = V(J) \quad \text{where}$$

$$J = I \cap k[x_1, \dots, x_m]$$

Further if G is a Gröbner basis for I w.r.t lex (or any elimination order) then

$G' = G \cap k[x_1, \dots, x_m]$ is a Gröbner basis for J

↑ $G = \{g_1(x_1, \dots, x_m), \dots, g_r(x_1, \dots, x_m), g_{r+1}(x_1, \dots, x_n), \dots, g_s(x_1, \dots, x_n)\}$

$G' = \{g_1, \dots, g_r\}$

$J = \langle g_1, \dots, g_r \rangle$ is called the elimination ideal.

Maps and Implicitization

Consider a map

$$\varphi : k^n \longrightarrow k^m$$
$$(p_1, \dots, p_n) \longmapsto (\varphi_1(p), \dots, \varphi_m(p))$$

for poly. $\varphi_1, \dots, \varphi_m$ in $k[t_1, \dots, t_n]$

$\boxed{Ex}$ $n=2$, $m=3$, $\varphi = (t_1, t_1 t_2, t_1 t_2^2)$

$$\overline{\varphi(k^2)} = V(x_1 x_3 - x_2^2)$$

Take $p = (0,0,1)$, $p \notin \varphi(k^2)$ as if

$(0,0,1) \neq (t_1, t_1 t_2, t_1 t_2^2)$ $\Rightarrow t_1 = 0 \Leftarrow$ $(0,0,0)$

but $p \in V(x_1 x_3 - x_2^2)$

Note: take $t_1 = \varepsilon^2$, $t_2 = \frac{1}{\varepsilon}$, $\varphi = (\varepsilon^2, \varepsilon, 1)$

$\therefore \lim_{\varepsilon \to 0} \varphi = (0,0,1)$.

$\boxed{Thm}$ Given a map $\varphi : k^n \to k^m$ as above

$I = \langle \varphi_1(t_1, \dots, t_n) - x_1, \dots, \varphi_m(t_1, \dots, t_n) - x_m \rangle \subseteq k[x_1, \dots, x_m, t_1, \dots, t_n]$

$\overline{\varphi(k^n)} = V(J)$ where $J = I \cap k[x_1, \dots, x_n]$

If $X \subseteq k^n$, $I_X = I(X)$, then

$$\overline{\varphi(X)} = V(J_X) \text{ for } J_X = (I + I_X) \cap k[x_1, \dots, x_m]$$

A toric variety is the closed image of a map defined by monomials.