The Geometry of Elimination

Projection maps for affine varieties

$V = V(f_1, \dots, f_s) \subseteq \mathbb{C}^n$. To eliminate the first l var.

Projection map

$$\pi_l : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-l}$$
$$(a_1, \dots, a_n) \longmapsto (a_{l+1}, \dots, a_n)$$

Note $\pi_l(V) \subseteq \mathbb{C}^{n-l}$

l^{th} elimination ideal

Lemma Let $I_l = (f_1, \dots, f_s) \cap \mathbb{C}[x_{l+1}, \dots, x_n]$

Then

$$\pi_l(V(f_1, \dots, f_s)) \subseteq V(I_l) \subseteq \mathbb{C}^{n-l}.$$

$I = (f_1, \dots, f_s)$

Proof: Fix $f \in I_l$. If $(a_1, \dots, a_n) \in V = V(f_1, \dots, f_s)$

$\Rightarrow$ f vanishes at $(a_1, \dots, a_n)$ since $f \in I$

But $f \in I_l$ $\Rightarrow$ only involves $x_{l+1}, \dots, x_n$

$\in k[x_1, \dots, x_n]$

$\therefore$ $f(0, \dots, 0, a_{l+1}, \dots, a_n) = f(\pi_l(a_1, \dots, a_n)) = 0$

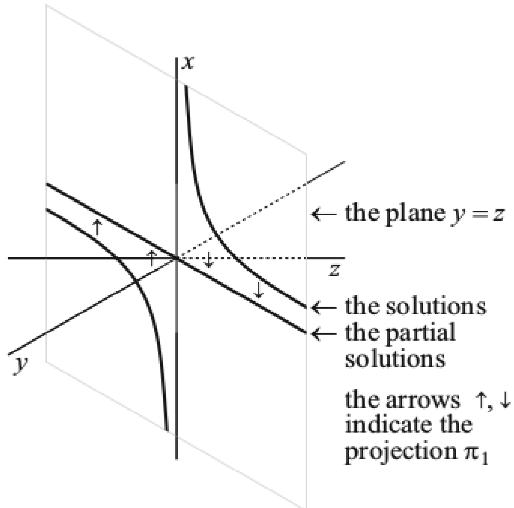
$\therefore$ $f(p) = 0$ $\forall p \in \pi_l(V)$ $\Rightarrow$ $\pi_l(V) \subseteq V(I_l)$ ∎

For $V = V(I)$.

$$\pi_l(V) = \left\{ (a_{l+1}, \dots, a_n) \in V(I_l) \;\middle|\; \exists\; a_1, \dots, a_l \in \mathbb{C} \text{ s.t. } (a_1, \dots, a_l, a_{l+1}, \dots, a_n) \in V(I) \right\}$$

all partial solutions which extend to complete solutions.

$I = (xy-1, xz-1)$



$I_1 = (y-z)$

$$\pi_1(V) = \{(a,a) \in \mathbb{C}^2 \mid a \neq 0\}$$

$\pi_1(V)$ is <u>not</u> an affine variety, $(0,0)$ is missing

Extension theorem (Geometrically)

<u>Theorem</u>) (Geo. Ext. Thm.)

Given $V = V(f_1, \ldots, f_s) \subseteq \mathbb{C}^n$, $c_i \in \mathbb{C}[x_2, \ldots, x_n]$ as in Extension thm. If I_1 is the first elim. ideal of $I = (f_1, \ldots, f_s)$ we have

only want part of $V(c_1, \ldots, c_s)$ that is in V

$$V(I_1) = \pi_1(V) \cup (V(c_1, \ldots, c_s) \cap V(I_1)) \subseteq \mathbb{C}^{n-1}$$

where $\pi_1 : \mathbb{C}^n \to \mathbb{C}^{n-1}$ is projection

Aside: Say instead

$$\pi_1(V) = V(I_1) \setminus V(c_1, \ldots, c_s)$$

<u>Proof (idea):</u> Translate ext. theorem to varieties

idea $\pi_1(V)$ & gets all points not in $V(c_1, \ldots, c_s)$

number $f_i = c_i(x_2, \dots, x_n)\, x_1^{N_i}$ ← largest x_1 exp.

But we may have points in $V(I_1)$ that are also in $V(c_1, \dots, c_s)$

taking the union $\pi_1(V)$ with $V(c_1, \dots, c_s)$ in $V(I_1)$ adds these.

$\boxtimes$

theorem says $\pi_1(V)$ "fill up" $V(I_1)$ except for part in $V(c_1, \dots, c_s)$

Ex) $V(xy-1, xz-1) = V(\overset{f_1}{(y-z)x^2 + yx - 1}, \overset{f_2}{(y-z)x^2 + xz - 1})$

$I_1 = (y-z)$

but with generators the coef. of highest x power if f_1, f_2 is

$c_1 = c_2 = y - z$

we lose all points

$V(c_1, c_2) = V(y-z) = V(I_1)$.

<u>Theorem 1</u> (The closure Theorem)

Let $V = V(f_1, \dots, f_s) \subseteq \mathbb{C}^n$ and I_ℓ ℓ^{th} elimination ideal of $I = (f_1, \dots, f_s)$. Then:

- $V(I_\ell)$ is the smallest affine variety containing $\pi_\ell(V) \subseteq \mathbb{C}^{n-\ell}$.
- When $V \neq \emptyset \Rightarrow \exists$ $W \subsetneq V(I_\ell)$ s.t. (a variety)

$$V(I_\ell) \setminus \underset{\substack{\uparrow \\ \text{i.e. } W = V(c_1,\dots,c_s)}}{W} \subseteq \pi_\ell(V)$$

Need Nullstellensatz.

$V(I_\ell)$ smallest means

- $\pi_\ell(V) \subseteq V(I_\ell)$
- If Z is any other affine variety in $\mathbb{C}^{n-\ell}$ containing $\pi_\ell(V) \Rightarrow V(I_\ell) \subseteq Z$.

i.e. No variety between $\pi_\ell(V)$ and $V(I_\ell)$.

We will (eventually) say $V(I_\ell)$ is the Zariski closure of $\pi_\ell(V)$.

$V(I_\ell)$ - closed set

$\pi_\ell(V)$ - can be open

varieties

$\exists\ Z_i \subseteq W_i \subseteq \mathbb{C}^{n-\ell} \quad 1 \leq i \leq m$

$$\pi_\ell(V) = \bigcup_{i=1}^{m} (W_i \setminus Z_i)$$

Corol Let $V = V(f_1,\dots,f_s) \subseteq \mathbb{C}^n$ and assume $\exists\ f_i$ s.t.

$$f_i = \underset{\substack{\uparrow \\ c_i \in K}}{c_i}\, x_1^{N_i} + \text{lower deg } x_1 \text{ terms}$$

Then $\pi_1(V) = V(I_1)$

Implicitization

find the <u>smallest variety</u> containing the parametrization (remember we can miss points when parametrizing)