Buchberger $\Leftrightarrow$ a basis $G = \{g_1, \dots, g_t\}$ of I is a GB iff $\overline{S(g_i, g_j)}^G = 0 \quad \forall i, j$

i.e. each S-poly

$$S(g_i, g_j) = \sum q_\ell g_\ell \text{ to using div}$$

Def Fix a mono. order let $G = \{g_1, \dots, g_t\} \subseteq k[x_1, \dots, x_n]$

$f \in k[x_1, \dots, x_n]$ reduces to zero modulo G

$f \to_G 0$ if f has a standard representation

$$f = A_1 g_1 + \dots + A_t g_t \quad , A_i \in k[x_1, \dots, x_n]$$

s.t. when $A_i g_i \neq 0$

$$\text{multdeg}(f) \geq \text{multdeg}(\underset{\neq 0}{A_i g_i}).$$

Lemma $G = \{g_1, \dots, g_t\} \subseteq k[x_1, \dots, x_n]$. Fix $f \in k[x_1, \dots, x_n]$

then $\overline{f}^G = 0$ implies $f \to_G 0$ however the converse is not true in general.

Proof: If $\overline{f}^G = 0 \Rightarrow f = q_1 g_1 + \dots + q_t g_t + 0$

$$\text{mult deg}(f) \geq \text{mult deg}(q_i g_i)$$

To see converse may fail:

$f = xy^2 - x \quad , G = (xy+1, y^2 - 1) \quad$ (lex)

$$xy^2 - x \to y(xy+1) + 0\cdot(y^2-1) + (-x-y)$$

So $\overline{f}^G = -x-y \neq 0$

But

$$xy^2 - x = 0\cdot(xy+1) + x(y^2-1) + 0$$

$$\text{multdeg}(xy^2-x) \geq \text{multdeg}(x(y^2-1))$$

$$\therefore f \to_G 0$$

Theorem

← Alterative Buchberger

A basis $G = \{g_1, \dots, g_t\}$ for an ideal I is a Gröbner basis iff $S(g_i, g_j) \xrightarrow[G]{} 0 \quad \forall i \neq j$.

Proof: $\Rightarrow$

If G is a GB $\Rightarrow \overline{S(g_i,g_j)}^G = 0 \Rightarrow S(g_i,g_j) \xrightarrow[G]{} 0$

$\Leftarrow$ In proof of original Buchberger crit we only required

$$S(g_i, g_j) = \sum A_\ell g_\ell$$

where $\text{multdeg}(S(g_i,g_j)) \geq \text{multdeg}(A_\ell g_\ell)$ $\cancel{x}0$

but this $S(g_i, g_j) \to_G 0$.

$\therefore$ GB iff all s-poly of pairs have a standard rep.

prop Given $G \subseteq K[x_1, \dots, x_n]$ finite suppose $f, g \in G$ s.t. $LM(f)$ and $LM(g)$ are relitively prime

$$\Rightarrow S(f,g) \to_G 0.$$

Proof: May suppose $LC(f) = LC(g) = 1$

$f = LM(f) + p \quad , \quad g = LM(g) + q$

since relitivley prime

$lcm(LM(f), LM(g)) \doteq LM(f)LM(g)$

$\therefore \quad S(f,g) = LM(g)\cdot f - LM(f)g$

$= (g-q)\cdot f - (f-p)\cdot g$

$= pg - qf$

If LM of two terms were the same

$\Rightarrow \quad LM(p)LM(g) = LM(q)LM(f)$

$\Rightarrow \quad LM(g) \mid LM(q)$

but $LM(g) > LM(q)$

since $g = LM(g) + q$

$\therefore \quad \text{multdeg}(S(f,g)) = \max(\text{multdeg}(p\cdot g), \text{multdeg}(qf))$

$\Rightarrow \quad S(f,g) \xrightarrow[G]{} 0$ since $f,g \in G$.

$\blacksquare$

Ex] $G = (yz+y, x^3+y, z^4)$ grlex. x^3, z^4 are relitivly prime

$\therefore \quad S(x^3+y, z^4) \xrightarrow[G]{} 0$ by Prop

but $\overline{S(x^3+y, z^4)}^G = y \neq 0$

One more...

$$LT(S(f,g)) < \operatorname{lcm}(LM(f), LM(g))$$

<u>Def:</u> Given $F = \{f_1, \dots, f_s\}$

$$S(f_i, f_j) = \sum A_\ell f_\ell$$

is a <u>lcm representation</u> provided that

$$\underline{\operatorname{lcm}(LM(f_i), LM(f_j)) > LT(A_\ell f_\ell).}$$

$\neq 0$

Ex] $f_1 = xz+1$, $f_2 = yz+1$, $f_3 = xz+y-z+1$, $x>y>z$

$$S(f_1, f_2) = \frac{\operatorname{lcm}(f_1,f_2)}{LT(f_1)} f_1 - \frac{\operatorname{lcm}(f_1,f_2)}{LT(f_2)} f_2 = \frac{xyz}{xz} f_1 - \frac{xyz}{yz} f_2$$

$$= xyz + y - xyz - x = -x+y$$

$$= -f_1 + 0 \cdot f_2 + f_3$$

$LT(S(f_1, f_3)) = y$

$LT(S(f_1,f_2))$

$\operatorname{multdeg}(f_3) > \operatorname{multdeg}(y)$

not standard rep

(since for st. rep. $\operatorname{multdeg}(S(f_i,f_j)) \geq \operatorname{multdeg}(A_\ell f_\ell)$)

is a p since

$$\operatorname{lcm}(LM(f_1), LM(f_3)) = xz > xz = LT(A_\ell f_\ell)$$

or

yz

Theorem A basis $G = (g_1, \dots, g_t)$ for I is a GB iff $S(g_i, g_j)$ has an lcm rep $\forall i \neq j$

Skip 2.10

Elimination

Def: Given $I = (f_1, \dots, f_s) \subseteq k[x_1, \dots, x_n]$ the ℓ^{th} elimination ideal in $k[x_{\ell+1}, \dots, x_n]$ is

$$I_\ell = I \cap k[x_{\ell+1}, \dots, x_n].$$