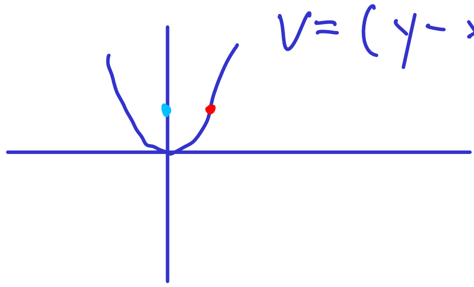
- Affine Space : k^n , $a = (a_1, \ldots, a_n) \in k^n$ $a_i \in k$
- variety is the vanishing set of a system of poly

- Every point in k^n is a variety

$$\{(a_1, \ldots, a_n)\} = V(x_1 - a_1, \ldots, x_n - a_n)$$

Ex]

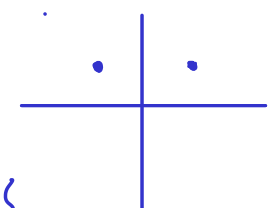


$V = (y - x^2)$

$(1,1) \in V$ Since $1 - 1^2 = 0$

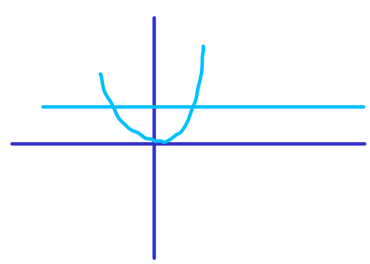
$(0,1) \notin V$ $1 - 0^2 \neq 0$

Intersections $V(y - x^2, y - 1)$



$\{(1,1), (-1,1)\}$

Unions $V((y-x^2)(y-1))$



$$I(V) = \{ f \in k[x_1, \ldots, x_n] \mid f(a) = 0 \ \forall a \in V \subseteq k^n \}$$

$(f_1, \ldots, f_s) \subseteq I(V(f_1, \ldots, f_s))$ i.e. $(x^2, y^2) \subsetneq I(V(x^2, y^2)) = (x, y)$

$V = W$ iff $I(V) = I(W)$

Ch 2.

Problems:

- Ideal desc: are all ideals fin. gen.?
- Ideal membership. $f \in (f_1, \dots, f_s)$

  $V(f) \supseteq V(f_1, \dots, f_s)$

  solved in $k[x]$ using div. alg. and gcd

- solutions $f_1 = \dots = f_s$ i.e. find $V(f_1, \dots, f_s)$

Ex]

$$f_1 = 2x_1 + 3x_2 - x_3 = 0$$
$$f_2 = x_1 + x_2 - 1 = 0$$
$$f_3 = x_1 + x_3 - 3 = 0$$

after row reduction

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

set $x_3 = t \in K$

$$x_1 = -t + 3$$
$$x_2 = t - 2$$
$$x_3 = t$$

$$V(f_1, f_2, f_3) = \{ (3-t, t-2, t) \mid t \in K \} \subseteq K^3$$

Monomial ordering for $K[x_1, \dots, x_n]$

Think of dividing $f(x) = x^5 - 3x^2 + 1$ by $g(x) = x^2 - 4x + 7$

i.e.

$$LT(f) = x^5 \quad , \quad LT(g) = x^2$$

$\therefore$ subtract $\frac{LT(f)}{LT(g)} g(x) = x^3 g(x)$ from $f(x)$

give
$$4x^4 - 7x^3 - 3x^2 + 1$$

key concept is ordering i.e.

$$x^{m+1} > x^m > \dots > x^2 > x > 1$$

in row reduction

$$x_1 > x_2 > \dots > x_n$$

i.e how do we order x^3yz and x^2y^2z

Note for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

$\therefore$ ordering monomials in $K[x_1, \dots, x_n] \Leftrightarrow$ ordering vectors $\alpha \in \mathbb{Z}^n_{\geq 0}$

want:

- Total ordering:
  - For every pair x^α, x^β exactly one of

Not in $k[x_1,\dots,x_n]$ as an ordering

$x^\alpha > x^\beta$ or $x^\alpha = x^\beta$ or $x^\beta > x^\alpha$

must be tre

  - Transitive, i.e. $x^\alpha > x^\beta$, $x^\beta > x^\gamma$

$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ $\Rightarrow$ $x^\alpha > x^\gamma$

For poly we have sums and products

- Sums are OK since we write largest monomials first
- Products:

-If $x^\alpha > x^\beta$ and x^γ is any monomial

$\Rightarrow$ $x^\alpha x^\gamma > x^\beta x^\gamma$

in $\mathbb{Z}^n_{\geq 0}$ $\alpha + \gamma > \beta + \gamma$ whenever $\alpha > \beta$

Summary

Def: A monomial ordering $>$ on $K[x_1,\dots,x_n]$ is a relation on $\mathbb{Z}^n_{\geq 0}$ $\Leftrightarrow$ on x^α, $\alpha \in \mathbb{Z}^n_{\geq 0}$ s.t:

- $>$ is a total ordering
- $\alpha > \beta \Rightarrow \alpha + \gamma > \beta + \gamma \quad \forall \gamma \in \mathbb{Z}^n_{\geq 0}$.
- $>$ is a well-ordering on $\mathbb{Z}^n_{\geq 0}$ i.e. every non-empty subset of $\mathbb{Z}^n_{\geq 0}$ has a least element

$A \subseteq \mathbb{Z}^n_{\geq 0}$ $\exists$ $\alpha \in A$ s.t $\beta > \alpha$ $\forall \beta \neq \alpha, \beta \in A$.

Lemma: $>$ on $\mathbb{Z}_{\geq 0}^n$ is a well ordering iff -

every sequence $\alpha(1) > \alpha(2) > \alpha(3) > \dots$

eventually terminates

<u>Proof:</u>

Contrapositive

$>$ is not a well ordering $\Leftrightarrow$ $\exists$ infinite strictly decreasing sequence

$\Rightarrow$

$\exists$ $S \subseteq \mathbb{Z}_{\geq 0}^n$ with no least element

Pick $\alpha(1) \in S$

$\alpha(1) > \alpha(2) > \alpha(3) > \dots$ infinitly decreasing

$\Leftarrow$ $\{\alpha(1), \alpha(2), \dots\}$

↑ no least element if

$\alpha(1) > \alpha(2) > \dots$ —.

usual numeric on $\mathbb{Z}_{\geq 0}$ $\quad m+1 > m \dots$

Def: 1 (Lexicographic order) $=$ Lex

$\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$

$\alpha >_{lex} \beta$ if the Left most non-zero entery of $\alpha - \beta$ is positive, $\Rightarrow$ $x^\alpha >_{lex} x^\beta$

Ex] which is Bigger in Lex?

$\boxed{(1,2,0)}$ This is bigger or $(0,3,4)$ $\quad (1,2,0)-(0,3,4)=(1,-1,-4)$

$xy^2 >_{lex} y^3z^4$

$\boxed{(3,2,4)}$ or $(3,2,1)$ $\quad (3,2,4)-(3,2,1) = (0,0,3)$

$x^3y^2z^4 >_{lex} x^3y^2z^1$

$(1,0,\dots,0) >_{lex} (0,1,\dots,0) >_{lex} \cdots >_{lex} (0,0,\dots,0,1)$

$x_1 >_{lex} x_2 >_{lex} \cdots >_{lex} x_n$

$x > y > z$