H.B Thm. + G.B.

Leading term Ideal of I = $(LT(I))$

Def) $I \subseteq K[x_1, \dots, x_n]$ an ideal, fix a mon. order.

set of all leading terms

- $LT(I) = \{ cx^\alpha \mid \exists f \in I \setminus \{0\} \text{ with } LT(f) = cx^\alpha \}$

- $(LT(I))$ – ideal generated by $LT(I)$.

$$I = (f_1, \dots, f_s)$$

$$(LT(f_1), \dots, LT(f_s)) \subseteq (LT(I))$$

May be different.

Ex) $I = (f_1, f_2)$, $f_1 = x^3 - 2xy$, $f_2 = x^2y - 2y^2 + 1$

use grlex

$$x f_2 - y f_1 = x^2$$

$\therefore x^2 \in I \quad \therefore x^2 = LT(x^2) \in (LT(I))$

But $\underset{x^3}{\underset{\shortparallel}{LT(f_1)}} \nmid x^2$, $\underset{x^2y}{\underset{\shortparallel}{LT(f_2)}} \nmid x^2$.

$\therefore x^2 \notin (LT(f_1), LT(f_2))$

Gröbner basis

Fix a monomial order

Def) A finite subset $G = \{g_1, \dots, g_t\}$ of an ideal $I \subseteq K[x_1,\dots,x_n]$ is a <u>Gröbner basis</u> if

$$(LT(g_1), \dots, LT(g_t)) = (LT(I)).$$

Prop: Let $I \overset{\neq 0}{\subseteq} K[x_1,\dots,x_n]$

- $(LT(I))$ is a monomial ideal
- There are $g_1, \dots, g_t \in I$ s.t. $(LT(I)) = (LT(g_1), \dots, LT(g_t))$ i.e. a GB exist for I.

proof:

- $(LM(g) \mid g \in I \setminus \{0\})$ is a mon. ideal.
  $= (LT(I))$.
- By Dickson $(LT(I)) = (LM(g_1), \dots, LM(g_t))$
  $= (LT(g_1), \dots, LT(g_t))$.

Thm] (Hilbert Basis Thm.) Every ideal $I \subset K[x_1,\dots,x_n]$ has a finite generating set $g_1,\dots,g_t \in I$ s.t. $I = (g_1,\dots,g_t)$.

Proof:

If $I = \{0\}$ then we are done

If I contains a nonzero poly. Construct a generating set $g_1, \dots, g_t$.

Fix a monomial order, the ideal

$$(LT(I)) = (LT(g_1), \dots, LT(g_t))$$ ← By Prop (this is a GB)

for some $g_1, \dots, g_t \in I$

Claim: $I = (g_1, \dots, g_t)$.

$(g_1, \dots, g_t) \subseteq I$ since $g_i \in I$.

Take f to be any poly in I, apply div. alg. to divide f by $g_1, \dots, g_t$

$$f = q_1 g_1 + \dots + q_t g_t + r$$

where no term in r is divisable by any of $LT(g_1), \dots, LT(g_t)$.

Our claim follows if $r = 0$

$$r = \underbrace{f - q_1 g_1 - \dots - q_t g_t}_{\in I} \quad \therefore r \in I$$

If $r \neq 0 \Rightarrow LT(r) \in (LT(I)) = (LT(g_1), \dots, LT(g_t))$

$\therefore$ by Lemma from monomial $LT(g_i) \mid LT(r)$ for some i.

but r is rem. of div $\therefore LT(g_i) \nmid LT(r)$

$\therefore$ Contradiction

$\therefore$ $r = 0$

$$\therefore \quad I = (g_1, \dots, g_t) \qquad \blacksquare$$

We have solved Ideal desc. problem with GB :)

"def" of GB

Eg] $\{g_1, \dots, g_t\} \subseteq I$ is a GB iff for all $f \in I$ $LT(g_i) \mid LT(f)$ for some i.

Cor] Any Grobner basis of I, say $\{g_1, \dots, g_t\}$ is also a basis for I, i.e.

$$I = (g_1, \dots, g_t)$$

App. H.B.T to prove Ascending Chain Condition

An Ascending Chain of ideals is

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots\dots$$

Ex]

$$(x_1) \subseteq (x_1, x_2) \subseteq \dots \subseteq (x_1, \dots, x_n) = I_n$$

Suppose we add an $f \in k[x_1, \dots, x_n]$

$(x_1, \dots, x_n, f)$

Either $f \in I_n \Rightarrow (x_1, \dots, x_n, f) = I_n$

If $f \notin (x_1, \dots, x_n) \Rightarrow (x_1, \dots, x_n, f) = K[x_1, \dots, x_n]$

any poly with constant term 0 is in I_n

Thm] (Ascending Chain Condition)

Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of ideals in $k[x_1, \dots, x_n]$. $\exists\ N \geq 1$ s.t.

$$I_N = I_{N+1} = I_{N+2} = \cdots$$

proof:

$$I = \bigcup_{i=1}^{\infty} I_i$$

By H.B.T. $I = (f_1, \dots, f_s)$

f_i must be in at least one I_j

Say $f_i \in I_{j_i}$ for some j_i, $i = 1, \dots, s$

$$N = \max \{ j_i \}$$

$\Rightarrow f_i \in I_N \ \forall i$ Since we have an ascending chain

$\therefore$ stabalize at $I_N = I_{N+1} = \cdots$ ∎

Def) Let $I \in k[x_1, \ldots, x_n]$ be an ideal

$$V(I) = \{(a_1, \ldots, a_n) \in k^n \mid f(a_1, \ldots, a_n) = 0 \ \forall f \in I\}$$

Variety of I

Prop) $V(I)$ is an affine variety. If $I = (f_1, \ldots, f_s)$ then $V(I) = V(f_1, \ldots, f_s)$.

obtain from H.B.T