17 in 8.2

Homogeneous Polynomials satisfy Euler's homogeneous function Formula

Let $f \in k[x_0, \dots, x_n]$ be homogeneous of degree d then

$$\sum_{i=0}^{n} x_i \frac{df}{dx_i} = d \cdot f$$

Ex) $f = x_0^3 - x_1 x_2^2$

$$x_0(3x_0^2) - x_1(x_2^2) - x_2(2x_1x_2)$$
$$= 3(x_0^3 - x_1 x_2^2)$$

Proof

Since f is homogenous of degree d

$$f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n)$$

$\Updownarrow$ By the chain Rule

$$\sum_{i=0}^{n} x_i \frac{df}{dx_i}(\lambda x_0, \dots, \lambda x_n) = d \lambda^{d-1} f(x_0, \dots, x_n)$$

$\lambda = 1$

$$\sum_{i=0}^{n} x_i \frac{df}{dx_i}(x_0, \dots, x_n) = d \cdot f(x_0, \dots, x_n)$$ ■

Recall in k^n we defined the Singular locus of $V = V(f) \subseteq k^n$ to be

$$V\left(f, \frac{df}{dx_1}, \dots, \frac{df}{dx_n}\right)$$

in $\mathbb{P}^n$, f homogeneous, the singular locus of $V = V(f) \subseteq \mathbb{P}^n$ is

$$V\left(f, \frac{df}{dx_0}, \ldots, \frac{df}{dx_n}\right) = V\left(\frac{df}{dx_0}, \ldots, \frac{df}{dx_n}\right)$$

[Ex_5.3]

Let k be any field, Suppose $I \subseteq k[x_1, \ldots, x_n]$ is s.t. $\dim_k(k[x_1,\ldots,x_n]/I) < \infty$. Then

$$\dim_k\left(k[x_1,\ldots,x_n]/\sqrt{I}\right) \leq \dim_k\left(k[x_1,\ldots,x_n]/I\right)$$

<u>Proof:</u> we know $I \subseteq \sqrt{I}$

$$k[x_1,\ldots,x_n]/I \simeq \operatorname{Span}(x^\alpha \mid x^\alpha \notin LT(I))$$

$$k[x_1,\ldots,x_n]/\sqrt{I} \simeq \operatorname{Span}(x^\alpha \mid x^\alpha \notin LT(\sqrt{I}))$$

$$LT(I) \subseteq LT(\sqrt{I}) \quad \text{Since } I \subseteq \sqrt{I}$$

Every basis vector x^α of $k[x_1,\ldots,x_n]/\sqrt{I}$ must appear in $k[x_1,\ldots,x_n]/I$ $\therefore$ # of basis vectors of $k[x_1,\ldots,x_n]/\sqrt{I}$ $\leq$ # basis vectors of $k[x_1,\ldots,x_n]/I$ ∎

Note $V = V(I) = V(\sqrt{I})$

points in $V \leq \dim_k\left(k[x_1,\ldots,x_n]/\sqrt{I}\right)$.

4.5 # 12

Let I be a proper ideal in $k[x_1, \ldots, x_n]$ (algebraically closed), then

$\sqrt{I}$ = intersection of all maximal ideals containing I.

Proof:

First show a 1-1 correspondence between points $a = (a_1, \ldots, a_n) \in V(I)$ and maximal ideals J s.t. $I \subseteq J$

Let $a \in V(I)$, know $a \in k^n \Leftrightarrow J = (x_1 - a_1, \ldots, x_n - a_n)$

WTS $I \subseteq J$

If $f \in I$ dividing f by $x_1 - a_1, \ldots, x_n - a_n$

$$f = q_1(x_1 - a_1) + \cdots + q_n(x_n - a_n) + c \quad c \in k$$

Since $a \in V(I)$

$$f(a) = 0 = q_1(a_1 - a_1) + \cdots + q_n(a_n - a_n) + c$$
$$= c$$
$$\therefore c = 0$$

$$\therefore f = q_1(x_1 - a_1) + \cdots + q_n(x_n - a_n) \in J.$$

$\therefore I \subseteq J$

For 1 point $a \in V(I)$ we have found

One maximal ideal $J = (x_1 - a_1, \dots, x_n - a_n)$ containing I.

Conversely, if J is any maximal ideal containing I then J has the form $J = (x_1 - b_1, \dots, x_n - b_n)$
$b = (b_1, \dots, b_n) \in k^n$

$$J \overset{1-1}{\longleftrightarrow} (b_1, \dots, b_n) \in k^n$$

Since $I \subseteq J$ if $f \in I$ same

$I \subseteq J$ OR
$\Rightarrow V(I) \supseteq V(J)$
$\therefore b \in V(I)$

$$\Rightarrow \quad f = h_1(x_1 - b_1) + \dots + h_n(x_1 - b_n)$$

$$f(b) = 0$$

$$\therefore \quad b \in V(I)$$

Now show

$$\sqrt{I} = I(V(I)) = \text{intersection of all maximal ideals containing } I$$
$$= \mathcal{M}$$

Let $f \in \mathcal{M}$

$a = (a_1, \dots, a_n) \in V(I)$

$\updownarrow$ 1-1

$I \subseteq J = (x_1 - a_1, \dots, x_n - a_n)$ maximal

$f \in J$, since J maximal and $\mathcal{M}$ is intersection

$\therefore \quad f = h_1(x_1 - a_1) + \cdots + h_n(x_n - a_n)$

$\Rightarrow f(a) = 0$

$$f \in I(V(I)) \quad \therefore \quad \mathfrak{m} \subseteq I(V(I))$$

Let $f \in I(V(I))$

Let $a \in V(I)$ divide f by $x_1 - a_1, \ldots, x_n - a_n$ as before

and since $f(a) = 0$

$c = 0$ as above

$\Rightarrow f = q_1(x_1 - a_1) + \cdots + q_n(x_n - a_n)$

$\Rightarrow f \in (x_1 - a_1, \ldots, x_n - a_n)$

which is a maximal ideal containing I

Since a is arbitrary and <u>every</u> $a \in V(I)$ has a corresponding maximal ideal containing I (and these are all the maximal ideals)

then $f \in \mathfrak{m} \quad \therefore \quad I(V(I)) \subseteq \mathfrak{m}$

$$\therefore \quad \sqrt{I} = I(V(I)) = \mathfrak{m}$$

∎