Final Topics

Primarily Chapters 4, 5, 8

No more than 1 question on ch 3

(#7 8.2)

Suppose that $f \in K[x_0, \dots, x_n]$ is <u>not</u> homogeneous but f vanishes at (<u>all</u> homogeneous coords of) $p \in \mathbb{P}^n$. Then each homogeneous component f_i of f must vanish at p.

K - infinite

Ex

$f = xy + z^2 + z^3 + xyz + x + y$

$f_1 = x + y$

$f_2 = xy + z^2$

$f_3 = xyz + z^3$

Proof

write f as

$$f = \sum_i \tilde{f}_i \qquad \deg(f_i) = i$$

homogeneous components

Set $p = (a_0 : \dots : a_n) \in \mathbb{P}^n$, $f(p) = 0$ ← we know

$= (\lambda a_0 : \dots : \lambda a_n) \quad \forall \lambda \neq 0$

$$f(\lambda a_0, \dots, \lambda a_n) = \sum f_i(\lambda a_0, \dots, \lambda a_n)$$

Every term in f_i has the form

$$c\, x_0^{\alpha_0} \cdots x_n^{\alpha_n}, \qquad \alpha_0 + \dots + \alpha_n = i$$

$$f_i(\lambda a_0, \dots, \lambda a_n) = \sum_{\alpha_0 + \dots + \alpha_n = i} c_\alpha (\lambda a_0)^{\alpha_0} \cdots (\lambda a_n)^{\alpha_n}$$

$$= \lambda^i f_i(a_0, \dots, a_n)$$

$\therefore$

$$f(\lambda a_0, \dots, \lambda a_n) = \sum_i \lambda^i f_i(a_0, \dots, a_n)$$

Consider

$$g(\lambda) = f(\lambda a_0, \dots, \lambda a_n)$$
$$= \sum_i \lambda^i f_i(a_0, \dots, a_n)$$

$$f(\lambda a_0, \dots, \lambda a_n) = 0 \quad \forall \ \lambda \neq 0 \in k$$

$$\Rightarrow \quad g(\lambda) = 0 \quad \forall \ \lambda \neq 0 \in k$$

k is infinite $\Rightarrow$ $g(\lambda) = 0$ the zero polynomial

$\therefore$ the coefficents of λ^i in $g(\lambda)$ are zero

$$\Rightarrow \quad f_i(a_0, \dots, a_n) = 0$$

$$\therefore \quad f_i(p) = 0 \quad \text{in } \mathbb{P}^n$$

$\blacksquare$

Consider the ideal $\boxed{I_0 = (x_0, \dots, x_n)} \subseteq K[x_0, \dots, x_n]$

(a) Show Every Proper homogeneous ideal in $K[x_0, \dots, x_n]$ is contained in I_0

(b) If I is a homogeneous ideal in $K[x_0, \dots, x_n]$, K alg. closed,

$V(I) = \phi \subseteq \mathbb{P}^n$ iff $I : (x_0, \dots, x_n)^\infty = K[x_0, \dots, x_n]$.

(a) Proof | Every homogeneous ideal I has a finite homogeneous basis $\Rightarrow$

$I = (f_1, \dots, f_s)$ with f_i homogeneous

Since I is proper ideal $f_i \neq c \in K$

$\therefore$ all f_i have no constant term

$$\therefore \quad f_i = \sum_{\alpha_0 + \dots + \alpha_n = d} c_\alpha x_0^{\alpha_0} \cdots x_n^{\alpha_n}, \quad d \geq 1$$

$\in (x_0, \dots, x_n)$

$\therefore \quad f_i \in I_0$ ∎.

$V(I_0) = V_p(x_0, \dots, x_n) = (0, \dots, 0) \notin \mathbb{P}^n$

$\Rightarrow \quad V(I_0) = \phi \subseteq \mathbb{P}^n$

Show $V(I) = \phi$ in $\mathbb{P}^n \iff I : (x_0, \dots, x_n)^\infty = k[x_0, \dots, x_n]$

Consider $V_a(I)$ in k^{n+1}

affine variety defined by the same homogeneous ideal

$V(I) = \phi \subseteq \mathbb{P}^n \iff V_a(I) \subseteq \{(0,\dots,0)\} \subseteq k^{n+1}$

projective variety

$\iff V_a(I) \setminus V_a(I_0) = \phi$

$\iff \overline{V_a(I) \setminus V_a(I_0)} = \phi$

affine result

$\iff V_a(I : I_0^\infty) = \phi$

$\iff I : I_0^\infty = k[x_0, \dots, x_n]$

$I : (x_0, \dots, x_n)^\infty = k[x_0, \dots, x_n]$.

$\blacksquare$

Radical Ideals, colon ideals, Saturation

(Ex. 4 in 4.4)

Let I, J be ideals in $k[x_1, \dots, x_n]$
Suppose I is radical. Then $I:J$ is radical
and $I:J = I:\sqrt{J} = I:J^\infty$

Proof)
Recall I is radical $\Leftrightarrow$ If $f^m \in I \Rightarrow f \in I$

$I:J = \{ f \in k[x_1, \dots, x_n] \mid fg \in I \ \forall g \in J \}$
$I:J^\infty = \{ f \in k[x_1, \dots, x_n] \mid \forall g \in J \ \exists m \geq 0 \text{ s.t } fg^m \in I \}$

First show $I:J$ is radical. Suppose $f^m \in I:J$
$m \geq 1$ (Otherwise $I:J = k[x_1, \dots, x_n]$)

$\therefore \ \forall g \in J \qquad f^m g \in I$

$\Rightarrow \quad f^m g \cdot g^{m-1} \in I$

$\Rightarrow \quad (fg)^m \in I \quad \Rightarrow \quad fg \in I$

$f \in I:J$

$\therefore \ I:J$ is radical

Show $I:J = I:\sqrt{J}$

$J \subseteq \sqrt{J}$

$\Rightarrow \ I:\sqrt{J} \subseteq I:J$

[consists of f s.t $fg \in I \ \forall g \in \sqrt{J}$
but $fg \in I \ \forall g \in J$]

Now show $I:J \subseteq I:\sqrt{J}$

Suppose $f \in I:J$ let $h \in \sqrt{J} \Rightarrow \exists m$ s.t

$h^m \in J$. Then

$f h^m \in I$ (since $f \in I:J$)

$\Rightarrow f^{m-1} f h^m \in I$

$\Rightarrow (fh)^m \in I$

$fh \in \sqrt{I} = I$ since I is radical

$\Rightarrow f \in I:\sqrt{J}$

$$I:J = I:\sqrt{J}$$

Show $I:J = I:J^\infty$

Know $I:J \subseteq I:J^\infty$

Show. $I:J^\infty \subseteq I:J$

$f \in I:J^\infty$

$\Rightarrow f g^m \in I$ for some $m \geq 0$ $\forall g \in J$

$(fg)^m \in I$

$\Rightarrow fg \in I$

$\Rightarrow f \in I:J$

$\overline{V(I) \setminus V(J)} = V(I:J)$ if I is radical.