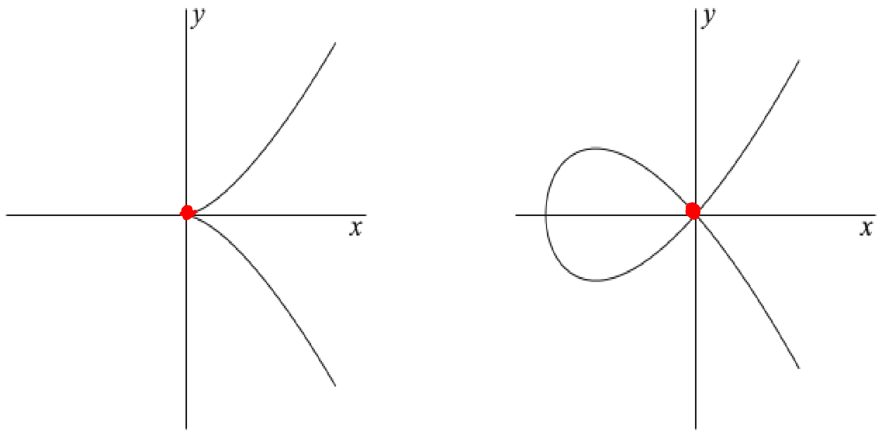
Singular Points on a Curve

- Plane curves in k^2 def. by $f(x,y)=0$, $f \in k[x,y]$
- Expect a well-defined tangent line a most points on curve
- This may fail



$y^2 = x^3$ $\qquad$ $y^2 = x^2(1+x)$

want tangent line to be unique and follow curve on both sides of a point

Consider $(a,b) \in V(f)$

L = line through (a,b) = $\begin{array}{l} x = a + ct \\ y = b + dt \end{array}$, $t \in K$

Def] m positive integer. $(a,b) \in V(f)$, L a line through (a,b). Then L meets $V(f)$ with multiplicity m at (a,b) if L can be parametrized so that $g(t) = f(a+ct, b+dt)$ has a root of multiplicity m at $t=0$.

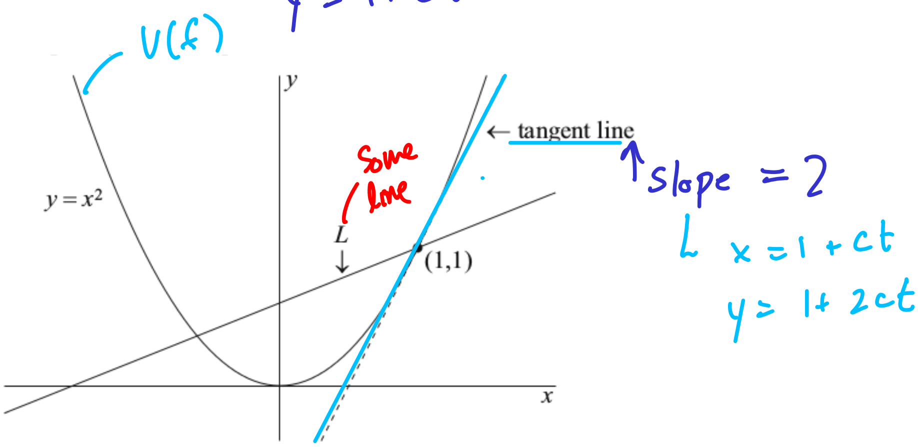
Rember $t=0$ is a root of mult. m of $g(t)$ if $g = t^m h(t)$ where $h(0) \neq 0$

Ex Let $f(x,y) = y - x^2$, $V(f)$ is parabola

$(1,1) \in V(f)$

a line L through $(1,1)$ of $V(f)$ is

$$x = 1 + ct$$
$$y = 1 + dt$$



Consider $g(t) = f(1+ct, 1+dt)$

$$= (1+dt) - (1+ct)^2 = t(-c^2 t + d - 2c)$$

when $d \neq 2c$ and $c \neq 0 \Rightarrow$ 2 distinct roots

$d \neq 2c$, $c = 0 \Rightarrow$ 1 distinct root

$\boxed{d = 2c}$ $g(t)$ has a root of mult. 2

$c \neq 0$

$\therefore$ Tangent line is where L meets $V(f)$ with mult 2 in this example

Generalize gradient vector

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

Prop Let $f \in K[x,y]$ $(a,b) \in V(f)$

(i) If $\nabla f(a,b) \neq (0,0)$ then $\exists$ a unique line through (a,b) which meets $V(f)$ with multiplicity ≥ 2.

(ii) If $\nabla f(a,b) = (0,0)$ then every line through (a,b) meets $V(f)$ with mult ≥ 2.

Proof: L - line through $(a,b) \in V(f)$

$$= \begin{cases} x = a + ct \\ y = b + dt \end{cases} \quad t \in K$$

$$g(t) = f(a+ct, b+dt)$$

$\therefore$ $t=0$ is a root of $g(t)$

In #5 on homework you will show

$t=0$ is a root of g with mult ≥ 2 iff $g'(0) = 0$

$$g'(t) = \frac{\partial f}{dx}(a+ct, b+dt) \cdot c + \frac{df}{dy}(a+ct, b+dt) \cdot d$$

$$g'(0) = \frac{df}{dx}(a,b) \cdot c + \frac{df}{dy}(a,b) \cdot d$$

If $\nabla f(a,b) = (0,0) \Rightarrow g'(0) = 0$ $\forall c,d$

$\therefore$ all lines L meet $V(f)$ with mult ≥ 2.

(Proves (ii))

Suppose $\nabla f(a,b) \neq (0,0)$

$$(*) \quad \frac{df}{dx}(a,b)\cdot c + \frac{df}{dy}(a,b)\cdot d = 0$$

at most 1 of these coeff. is 0

$\therefore$ have a 1. dim solution space, i.e. a line

if $(\hat{c},\hat{d})$ are a solution all other solutions

$$(c,d) = \lambda(\hat{c},\hat{d}) \quad \lambda \in K.$$

i.e. all (c,d) that solve $(*)$ lie on the same line.

$\therefore$ $\exists$ a unique line L for which $g'(0)=0$
i.e. for which L meets $V(f)$ at (a,b) with mult. ≥ 2.

$\blacksquare$

Def Let $f \in K[x,y]$, $(a,b) \in V(f)$

- If $\nabla f(a,b) \neq (0,0)$ then the tangent line to $V(f)$ at (a,b) is the unique line through (a,b) which meets $V(f)$ with mult ≥ 2.
  Call (a,b) a smooth or a nonsingular (or regular) point of $V(f)$

- If $\nabla f(a,b) = (0,0) \Rightarrow (a,b)$ is a singular point of $V(f)$.

(with $\frac{d}{dx}(x^n y^m) := n x^{n-1} y^m$ etc)

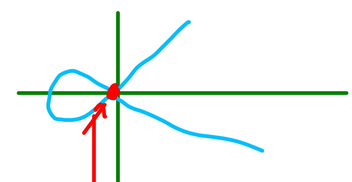
More generally a point $p \in V$ a variety in k^n, $V = V(f_1, \dots, f_m)$ will be a singular point if:

$$J(V) = \begin{bmatrix} \frac{df_1}{dx_1} & \cdots & \frac{df_1}{dx_n} \\ \vdots & & \\ \frac{df_m}{dx_1} & \cdots & \frac{df_m}{dx_n} \end{bmatrix}$$

has rank less then $\min(m,n)$, i.e. less then maximal/full rank.

Note we can also compute all singular points in a curve $V \doteq V(f)$ i.e.

$$\underbrace{V_{sing}}_{\text{singular locus}} = V(f) \cap V(\nabla f)$$



Ex) $f(x,y) = y^2 - x^2(1+x)$, $V = V(f)$

$$V_{sing} = V(y - x^2 - x^3, -2x - 3x^2, 2y)$$

Computing a lex GB

$$V_{sing} \doteq V(x,y) \Rightarrow V_{sing} = \{(0,0)\}$$ only Singularity.

Note we have have shown ($\mathbb{R}$) that ∇f is perpendicular to the tangent line of $V(f)$ at (a,b)

$$\text{Since } \nabla f(a,b) \cdot (c,d) = 0$$