Implicitization

find the smallest variety containing the parametrization (remember we can miss points when parametrizing)

Suppose we find smallest V

Q: • does a parametrization fill up all of V?

• if missing points, how do we find them

A: GB + Extension Thm.

[Ex]

$$x = t+u$$
$$y = t^2 + 2tu$$
$$z = t^3 + 3tu$$

(Tangent surface to twisted cubic)

?

General setup

$$x_1 = f_1(t_1, \dots, t_m) \in k[t_1, \dots, t_m]$$
$$\vdots$$
$$x_n = f_n(t_1, \dots, t_m)$$

geometrically

$$F : k^m \longrightarrow k^n$$
$$: (t_1, \dots, t_m) \longmapsto (f_1(t_1, \dots, t_m), \dots, f_n(t_1, \dots, t_m))$$

subset of k^n

– may not be an affine variety

Find smallest V s.t. $F(k^m) \subseteq V$

Do this by elimination

$$V = V(x_1 - f_1, \ldots, x_n - f_n) \subseteq k^{m+n}$$

$\updownarrow$

points are

$$(t_1, \ldots, t_m, f_1(t_1, \ldots, t_m), \ldots, f_n(t_1, \ldots, t_m))$$

$\therefore$ V is the graph of the function F.

$$i : k^m \to k^{m+n}$$
$$: (t_1, \ldots, t_m) \mapsto (t_1, \ldots, t_m, f_1(t_1, \ldots, t_m), \ldots, f_n(t_-))$$

$$\pi_m : k^{m+n} \to k^n$$
$$: (t_1, \ldots, t_m, x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n)$$

$$\begin{tikzcd} & k^{m+n} \arrow[dr, "\pi_m"] & \\ k^m \arrow[ur, "\text{graph } i"] \arrow[rr, "F", "\text{parametrization}"'] & & k^n \end{tikzcd}$$

$$i(k^m) = V$$

$$F(k^m) = \pi_m(i(k^m)) = \pi_m(V)$$

image of parametrization = projection of its graph

Theorem 1 (poly. Implicitization) — $x_1 = f_1(t_1,\dots,t_m)$ etc.

k infinite field. $F: k^m \to k^n$ function given by parametrization

Let $I = (x_1 - f_1, \dots, x_n - f_n) \subseteq k[t_1,\dots,t_m,x_1,\dots,x_n]$

$I_m = I \cap k[x_1,\dots,x_n]$ the m^{th} elim. ideal

Then $V(I_m)$ is the smallest variety in k^n containing $F(k^m)$ (i.e. $V(I_m)$ is the Zariski closure of $F(k^m)$ in k^n)

Proof:

$V = V(I)$

$F(k^m) = \pi_m(V)$

By Lemma image of proj π_m is contained in $V(I_m)$

$$F(k^m) = \pi_m(V) \subseteq V(I_m)$$

↑ show smallest

Suppose $h \in k[x_1,\dots,x_n]$ vanishes on $F(k^m)$

Can think of $h \in k[t_1,\dots,t_m,x_1,\dots,x_n]$ divide h by $x_1 - f_1, \dots, x_n - f_n$ using lex with $x_1 > \dots > x_n > t_1 > \dots > t_m$ gives

Since $LT(x_j - f_j) = x_j$

$$h(x_1,\dots,x_n) = q_1 \cdot (x_1 - f_1) + \dots + q_n \cdot (x_n - f_n) + r(t_1,\dots,t_m)$$

[and rem must not be divisable by any x_j]

Given any $a = (a_1,\dots,a_m) \in k^m$ we substitute $t_i = a_i$ $x_i = f(a)$

[h vanishes on $F(k^m)$]

$$0 = h(f_1(a), \dots, f_n(a)) = \underset{\substack{\downarrow \\ q_1(f_1(a)-f_1(a))}}{0} + \dots + 0 + r(a)$$

$\uparrow$ h vanishes on $F(k^m)$

$r(a) = 0 \quad \forall a \in k^m$ since k is infinite

$\Rightarrow r = \underset{\text{zero poly}}{0} \in k[t_1, \dots, t_m]$

$$\therefore \underbrace{h(x_1, \dots, x_n)}_{\in k[x_1, \dots, x_n]} = \underbrace{q_1(x_1 - f_1) + \dots + q_n(x_n - f_n)}_{\in I}$$

$\therefore h \in I$ and $h \in k[x_1, \dots, x_n]$

$\therefore h \in I \cap k[x_1, \dots, x_m] = I_m$

If $Z = V(h_1, \dots, h_s) \subseteq k^n$ s.t. $F(k^m) \subseteq Z$

$\Rightarrow h_i$ vanish on $F(k^m) \Rightarrow h_i \in I_m$

$\Rightarrow (h_1, \dots, h_s) \subseteq I_m$

$V(I_m) \subseteq V(h_1, \dots, h_s) = Z$

$\therefore V(I_m)$ is the smallest variety containing $F(k^m)$ ∎

Implicitization Alg

If $x_i = f_i(t_1, \dots, t_m)$, $f_1, \dots, f_n \in k[t_1, \dots, t_m]$

- Let $I = (x_1 - f_1, \dots, x_n - f_n)$
- Compute GB with lex $t_1 > \dots > t_m > x_1 > \dots > x_n$
- By Elim. Thm. poly with no t's

form a basis of I_m

- $V(I_m)$ = smallest variety containing the image of parametrization.

What about rational parametrizations

map F
$$\begin{cases} x_1 = \dfrac{f_1(t_1,\dots,t_m)}{g_1(t_1,\dots,t_m)} \\ \vdots \\ x_n = \dfrac{f_n(t_1,\dots,t_m)}{g_n(t_1,\dots,t_m)} \end{cases}$$

$f_i, g_i \in k[t_1,\dots,t_m]$

$F : k^m \to k^n$ may not be defined on all k^m

i.e. in $V(g_1 \cdots g_n)$ — union of $V(g_i)$'s $= \bigcup_i V(g_i)$

we have issues

Let $W = V(g_1 \cdots g_n) \subset k^m$

$$F(t_1,\dots,t_m) = \left(\frac{f_1}{g_1}, \dots, \frac{f_n}{g_n} \right)$$

defines a map $F : k^m \setminus W \to k^n$

want smallest variety in k^n containing $F(k^m)$.