The Zariski Topology

Def If V is a variety a subset $S \subseteq V$ is **Zariski dense** in V if $V = \overline{S}$

- **closed sets** are varieties, i.e. $V(f_1, \dots, f_s) = V(I)$ are closed
- open sets are formed by complements of closed sets i.e. $k^n \setminus V(I)$
  ↑ Distinguished open sets
- open sets $k^n \setminus V(I)$ are dense in k^n :
- $\pi_\ell(V)$ is Zariski dense in $V(I_\ell)$ (for alg. closed k)

Ex) $V = V(xz, yz)$, $W = V(z)$

↑ union x-y plane and z-axis (for V)

↑ x-y Plane (for W)

$V \setminus W = z\text{-axis} - (0,0,0)$

$\overline{V \setminus W} = z\text{-axis}$

↑ Can we do this with ideals?

Def: I, J ideals in $K[x_1,\dots,x_n]$

$$I:J = \{ f \in K[x_1,\dots,x_n] \mid f\cdot g \in I \quad \forall g \in J \}$$

↑ Note f may not be in I

ideal quotient or colon ideal

Prop: $I:J$ is an ideal which contains I

· $f \in I:J$, $h \in K[x_1,\dots,x_n] \Rightarrow f\cdot g \in I \quad \forall g \in J$

$h\cdot fg \in I$

since I is an ideal

$\therefore hf \in I:J$

Prop: I, J ideals in $K[x_1,\dots,x_n]$

(i) $V(I) = V(I+J) \cup V(I:J)$

(ii) V, W varieties

$$V = (V \cap W) \cup \overline{(V \setminus W)} \leftarrow$$

(iii) $\overline{V(I) \setminus V(J)} \subseteq V(I:J)$

proof: (ii) $V \setminus W \subseteq V \Rightarrow \overline{V \setminus W} \subseteq V$

and $V \cap W \subseteq V$

$$\Rightarrow (V \cap W) \cup \overline{(V \setminus W)} \subseteq V$$

$$V = V \cap W \cup (V \setminus W) \subseteq V \cap W \cup \overline{(V \setminus W)}$$

$$V = V \cap W \cup \overline{(V \setminus W)}$$

(iii) Show $I:J \subseteq I(V(I)\setminus V(J))$

Let $f \in I:J$, $a \in V(I)\setminus V(J)$

$\Rightarrow fg \in I \quad \forall g \in J$, $a \in V(I)$

$\Rightarrow f(a)g(a) = 0 \quad \forall g \in J$

$a \notin V(J)$

$\therefore \exists g \in J$ s.t. $g(a) \neq 0$

$\Rightarrow f(a) = 0 \quad \forall a \in V(I)\setminus V(J)$

$\therefore f \in I(V(I)\setminus V(J))$

$I:J \subseteq I(V(I)\setminus V(J))$

$V(I:J) \supseteq V(I(V(I)\setminus V(J)))$
$= \overline{V(I)\setminus V(J)}$.

(i) by (ii)

$V(I) = V(I+J) \cup \overline{V(J)\setminus V(J)}$

by (iii) $\subseteq V(I+J) \cup V(I:J)$.

$I \subseteq I+J$, $I \subseteq I:J$

$V(I+J) \subseteq V(I)$, $V(I:J) \subseteq V(I)$

$\therefore V(I+J) \cup V(I:J) \subseteq V(I)$ ∎

So do we have

$$\overline{V(I)\setminus V(J)} \subseteq V(I:J)$$

↑ equal? No ☹

not even over $\mathbb{C}$

$$I = (x^2(y-1)), \quad J = (x)$$

$$V(I) = V(x) \cup V(y-1) = V(J) \cup V(y-1)$$

$$\therefore \overline{V(I)\setminus V(J)} = V(y-1).$$

$$(x^2(y-1)) : (x) = (x(y-1))$$

$$V(x(y-1)) \neq V(y-1)$$

$$V(I:J^2) = V(y-1) = \overline{V(I)\setminus V(J)}$$

Def: I, J are ideals in $k[x_1,\dots,x_n]$ then the <u>saturation</u> of I w.r.t J is

$$I:J^\infty = \{ f \in k[x_1,\dots,x_n] \mid \forall g \in J, \exists N \geq 0 \text{ s.t. } fg^N \in I \}$$

Prop) I, J ideals in $k[x_1,\dots,x_n]$. $I:J^\infty$ is an ideal. Also:

(i) $I \subseteq I:J \subseteq I:J^\infty$

(ii) $I:J^\infty = I:J^N$ for $N > m$ for some m.

(iii) $\sqrt{I:J^\infty} = \sqrt{I}:J$

Proof: $J^{N+1} \subseteq J^N \Rightarrow I:J^N \subseteq I:J^{N+1}$

↑ less g's to require $f \cdot g \in I$

$\Rightarrow$ $I \subseteq I:J \subseteq I:J^2 \subseteq \cdots\cdots$

By the ACC $\exists N$ s.t. $I:J^N = I:J^{N+1}$

Show $I:J^\infty = I:J^N$

- If $f \in I:J^N$, $g \in J \Rightarrow g^N \in J^N$

  $\Rightarrow f g^N \in I$ for some $N \Rightarrow f \in I:J^\infty$

  $I:J^N \subseteq I:J^\infty$

- $f \in I:J^\infty$, $J = (g_1,\dots,g_s)$

  $\Rightarrow f g_i^{m_i} \in I \quad \forall i$

  $M = \max(m_i)$

  $\Rightarrow f g_i^M \in I \quad \forall i$

$J^{sM} \subseteq (g_1^M, \dots, g_s^M)$

$\Rightarrow \quad f J^{sm} \subseteq I$

$\uparrow$ each term has g_i^m

$\Rightarrow \quad f \in I : J^{sm}$