$$I_1 \subseteq I_2$$

$$V(I_2) \subseteq V(I_1)$$

$$I_1 = (x) \quad , \quad I_2 = (x, y)$$

<u>Algorithm to compute intersection of ideals</u>

Input : $I = (f_1, \dots, f_r)$, $J = (g_1, \dots, g_s)$ ideals $k[x_1, \dots, x_n]$

- $D = tI + (1-t)J = (tf_1, \dots, tf_r, (1-t)g_1, \dots, (1-t)g_s)$
- Compute a lex GB of D
- Elements of GB with no t's for a GB for $I \cap J$.

<u>Def</u> $f, g \in k[x_1, \dots, x_n]$

$h = \mathrm{lcm}(f, g)$

- $f \mid h$, $g \mid h$
- If $f \mid P$, $g \mid P \Rightarrow h \mid P$

Ex] $\mathrm{lcm}(x^2y, xy^2) = x^2y^2$

more generally if

$$f = c\, f_1^{a_1} \cdots f_r^{a_r} \quad , g = \tilde{c}\, g_1^{b_1} \cdots g_s^{b_s}$$

irr. factors

- It could be that $f_i = \check{c}\, g_i$

$$\mathrm{lcm}(f, g) = f_1^{\max(a_1, b_1)} \cdots f_\ell^{\max(a_\ell, b_\ell)} \cdot g^{b_{\ell+1}} \cdots g^{b_s} f_{\ell+1}^{a_{\ell+1}} \cdots f_r^{a_r}$$

if f, g have no common factors

$$\mathrm{lcm}(f, g) = fg$$

Prop $I = (f)$, $J = (g)$

$$I \cap J = (\mathrm{lcm}(f, g))$$

This gives an Algorithm to compute $\mathrm{lcm}(f, g)$ (without factoring) i.e. compute $(f) \cap (g)$

Note

$$\mathrm{lcm}(f, g) \cdot \gcd(f, g) = f \cdot g$$

$\therefore$ our lcm algorithm gives a gcd algorithm

Thm If I, J are ideals in $k[x_1, \dots, x_n]$ then $V(I \cap J) = V(I) \cup V(J)$

$= V(I \cdot J)$

Proof $a \in V(I) \cup V(J)$

$\Rightarrow$ $f(a) = 0$ $\forall f$ in either I or J

$\Rightarrow$ $f(a) = 0$ $\forall f \in I \cap J$

$\Rightarrow a \in V(I \cap J)$

$\Rightarrow V(I) \cup V(J) \subseteq V(I \cap J)$

On the other hand

$IJ \subseteq I \cap J$

$\Rightarrow$ $V(I \cap J) \subseteq V(IJ) = V(I) \cup V(J)$ ∎

$$V(IJ) = V(I \cap J)$$

Prop] I, J ideals

$$\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$$

Proof: $f \in \sqrt{I \cap J}$ $\Rightarrow$ $f^m \in I \cap J$

$\therefore$ $f^m \in I$, $f^m \in J$

$\Rightarrow f \in \sqrt{I}$ and $f \in \sqrt{J}$

$\Rightarrow$ $\sqrt{I \cap J} \subseteq \sqrt{I} \cap \sqrt{J}$

$f \in \sqrt{I} \cap \sqrt{J}$ $\Rightarrow$ $f^m \in I$, $f^p \in J$

$\therefore$ $f^m f^p \in I \cap J$

$\quad \| \quad f^{m+p}$

$\Rightarrow$ $f \in \sqrt{I \cap J}$ ∎

Zariski closures

suppose $S \subseteq k^n$ is a set (not nessusarily a variety)

$$I(S) = \{ f \in k[x_1, \dots, x_n] \mid f(a) = 0 \ \forall a \in S \}$$

this is a radical ideal

By the same proof as case $S = V$ a variety

$\therefore$ $V(I(S))$ is a variety which contains S

Prop $S \subseteq k^n$ ← set. $V(I(S))$ is the smallest variety which contains S, i.e. if $S \subseteq W$ for a variety W $\Rightarrow$ $V(I(S)) \subseteq W$

Proof: If $W \supseteq S \Rightarrow I(W) \subseteq I(S)$

$$\Rightarrow \quad V(I(S)) \subseteq V(I(W)) = W \quad \blacksquare$$

Def The Zariski closure of a subset $S \subseteq k^n$ is the smallest affine variety containg S that is

$$\overline{S} = V(I(S))$$

↑ zariski closure

Lemma Let S, T be subsets of k^n. Then

- $I(\overline{S}) = I(S)$
- $S \subseteq T \Rightarrow \overline{S} \subseteq \overline{T}$
- $\overline{S \cup T} = \overline{S} \cup \overline{T}$

Thm (closure Thm, part #1)

k alg. closed. $V = V(f_1, \dots, f_s) \subseteq k^n$

$\pi_\ell : k^n \longrightarrow k^{n-\ell}$ projection onto last $n-\ell$ coords. $I_\ell = (f_1, \dots, f_s) \cap k[x_{\ell+1}, \dots, x_n]$

ℓ^{th} elimination ideal.

then $V(I_\ell) = \overline{\pi_\ell(V)}$ Zariski closure

Proof: Show $V(I_\ell) = V(I(\pi_\ell(V)))$

By a lemma from 3.2

$$\pi_\ell(V) \subseteq V(I_\ell)$$

$\overline{\pi_\ell(V)} = V(I(\pi_\ell(V)))$ is the smallest variety containing $\pi_\ell(V)$

$$\therefore \quad \overline{\pi_\ell(V)} \subseteq V(I_\ell)$$

Now suppose $f \in I(\pi_\ell(V))$

i.e $\quad f(a_{l+1}, \dots, a_n) = 0 \quad \forall (a_{l+1}, \dots, a_n) \in \pi_l(V)$

moving f to $k[x_1, \dots, x_n]$

$$f(a_1, \dots, a_n) = 0 \quad \forall (a_1, \dots, a_n) \in V$$

$\uparrow$ since $x_1, \dots, x_l$ don't appear in f.

$$\Rightarrow f \in I(V)$$

By Nullstellensatz $\Rightarrow f \in \sqrt{(f_1, \dots, f_s)}$

$$f^N \in (f_1, \dots, f_s) \quad \text{for some } N$$

$$\therefore \quad f^N \in (f_1, \dots, f_s) \cap k[x_{l+1}, \dots, x_n] = I_l$$

$$\Rightarrow f \in \sqrt{I_l}$$

$$I(\pi_l(V)) \subseteq \sqrt{I_l}$$

$$V(I_l) \subseteq V(I(\pi_l(V))) = \overline{\pi_l(V)}$$

$\blacksquare$