Prop: Suppose K is a field containg $\mathbb{Q}$.

Let $I = (f) \subseteq K[x_1, \dots, x_n]$. Then $\sqrt{I} = (f_{red})$ where

$$f_{red} = \frac{f}{\gcd\left(f, \frac{df}{dx_1}, \dots, \frac{df}{dx_n}\right)}$$

Proof: we know $\sqrt{(f)} = (f_1 \cdots f_r)$

where $f = f_1^{a_1} \cdots f_r^{a_r}$

show

$$\gcd\left(f, \frac{df}{dx_1}, \dots, \frac{df}{dx_n}\right) = f_1^{a_1 - 1} \cdots f_r^{a_r - 1}$$

$= \gcd(\sim)$

$f = f_1^{a_1} \cdots f_r^{a_r}$

By product rule

$$\frac{df}{dx_j} = f_1^{a_1 - 1} \cdots f_r^{a_r - 1}\left(a_1 \frac{df_1}{dx_j} f_2 \cdots f_n + \cdots a_r f_1 \cdots f_{r-1} \frac{df}{dx_j}\right)$$

$$\Rightarrow \quad f_1^{a_1 - 1} \cdots f_r^{a_r - 1} \mid \gcd(\sim)$$

Show $\forall i \; \exists \; \frac{\partial f}{dx_j}$ which is not divisible by $f_i^{a_i}$

(this would mean anything bigger couldn't be gcd since f_i's are irr.)

write $f = f_i^{a_i} h_i$ when $f_i \nmid h_i$

f is non-constant

$\frac{df}{dx_j} \neq 0$ for some j

$$\frac{df}{dx_j} = f_i^{a_i - 1}\left(a_i \frac{df_i}{dx_j} h_i + f_i \frac{dh_i}{dx_j} \right)$$

If this is divisible by $f_i^{a_i}$

$$f_i \mid \frac{df_i}{dx_j} h_i$$

$\Rightarrow$ Since $f_i \nmid h_i$

$$\Rightarrow f_i \mid \frac{df_i}{dx_j}$$

But since $\mathbb{Q} \subseteq K \Rightarrow \left(\frac{df_i}{dx_j} \neq 0 \right)$

and $\frac{df}{dx_j}$ has smaller degree than f_i

$$\therefore f_i^{a_i} \nmid \frac{df}{dx_i}$$

$$\therefore \gcd\left(f, \frac{df}{dx_1}, \ldots, \frac{df}{dx_n}\right) = f_1^{a_1 - 1} \cdots f_r^{a_r - 1}$$

$\blacksquare$

<u>Def</u>: I, J ideals in $k[x_1, \ldots, x_n]$ then the sum of I and J is

$$I + J = \{ f + g \mid f \in I, g \in J \}$$

$$I \cdot J = \{ f_1 g_1 + \cdots + f_r g_r \mid f_1, \ldots, f_r \in I, g_1, \ldots, g_r \in J \}$$

$I + J$, $I \cdot J$ are ideals

Prop $I = (f_1, \dots, f_r)$, $J = (g_1, \dots, g_s)$

$I + J$ is the smallest ideal containing I and J

$$I + J = (f_1, \dots, f_r, g_1, \dots, g_s)$$

$$(f_1, \dots, f_r) = (f_1) + \cdots + (f_r)$$

i.e. sums of ideals correspond to intersections of varieties

i.e. $V(f_1, \dots, f_r) = V(f_1) \cap \cdots \cap V(f_r)$

Thm I, J are ideals in $k[x_1, \dots, x_n]$

Then $V(I + J) = V(I) \cap V(J)$.

Prop $I = (f_1, \dots, f_r)$, $J = (g_1, \dots, g_s)$

$$I \cdot J = (f_i g_j \mid 1 \leq i \leq r, 1 \leq j \leq s)$$

Products of ideals correspond geometrically to unions

i.e.

$$V(f_1, \dots, f_r) \cup V(g_1, \dots, g_s) = V(f_i g_j \mid 1 \leq i \leq r, 1 \leq j \leq s)$$

Thm I, J are ideals in $k[x_1, \dots, x_n]$ then

$$V(I \cdot J) = V(I) \cup V(J).$$

Ex) $V(xz, yz) = V((x,y)\cdot(z)) = V(x,y) \cup V(z)$

Intersection of ideals

Prop/Def: If I, J are ideals, $I \cap J$ is also an ideal

Thm) Let $\underset{= (f_1, \dots, f_r)}{I}, J$ be ideals in $k[x_1, \dots, x_n]$

in $k[x_1, \dots, x_n, t]$

Then $I \cap J = (\underset{= (tf_1, \dots, tf_r)}{tI} + (1-t)J) \cap k[x_1, \dots, x_n]$

Proof:

- $tI + (1-t)J$ is ideal in $k[x_1, \dots, x_n, t]$

Show $I \cap J \subseteq (tI + (1-t)J) \cap k[x_1, \dots, x_n]$:

- $f \in I \cap J$

$\Rightarrow f \in I \Rightarrow tf \in tI \subseteq k[x_1, \dots, x_n, t]$

and $f \in J \Rightarrow (1-t)f \in (1-t)J \subseteq k[x_1, \dots, x_n, t]$

$\therefore f = tf + (1-t)f \in tI + (1-t)J$

$\therefore f \in (tI + (1-t)J) \cap k[x_1, \dots, x_n]$

Now show $(tI + (1-t)J) \cap k[x_1, \dots, x_n] \subseteq I \cap J$:

$f \in (tI + (1-t)J) \cap k[x_1, \dots, x_n]$

$\Rightarrow f(x) = g(x,t) + h(x,t)$ $\quad g \in tI, \; h \in (1-t)J$

Set $t=0$

$\Rightarrow \quad g(x, 0) = 0$ since $g \in tI$ and $t=0$

$\therefore \quad f(x) = h(x,0) \in (1-0)\cdot J = J$

$\therefore \quad f \in J$

Set $t=1$

$\Rightarrow \quad h(x,1) = 0$ since $h \in (1-t)J$

$\therefore \quad f(x) = g(x,1) \in 1 \cdot I = I \quad \therefore \quad f(x) \in I \cap J.$

$\blacksquare$