when does

$$f_1 = 0$$
$$\vdots$$
$$f_s = 0 \qquad \text{have a solution}$$

$V(f_1, \dots, f_s) = \emptyset$ iff $1 \in (f_1, \dots, f_s)$

Empty var Alg.

Input : ideal I

- Compute a reduced GB G of I
- If $G = \{1\} \Rightarrow V(I) = \emptyset$

  otherwise $V(I) \neq \emptyset$.

Ex $I_1 = (x^n, y^m)$, $I_2 = (x, y)$

$$V(I_1) = V(I_2) = \{(0,0)\}$$

Thm (Hilberts Nullstellensatz). Let k be an alg. closed field. If $f, f_1, \dots, f_s \in k[x_1, \dots, x_n]$ then $f \in I(V(f_1, \dots, f_s))$ if and only if

$$f^m \in (f_1, \dots, f_s)$$

for some $m \geq 1$, $m \in \mathbb{Z}$.

Proof: Given $f \neq 0$ s.t. $f(p) = 0 \quad \forall p \in V(f_1, \dots, f_s)$

we must show $\exists \; m \geq 1$ s.t

$$f^m = \sum_{i=1}^{s} A_i f_i \qquad A_i \in k[x_1, \dots, x_n]$$

Use "Rabinowitsch trick": Consider

$$\tilde{I} = (f_1, \dots, f_s, 1 - yf) \subseteq k[x_1, \dots, x_n, y]$$

Show $V(\tilde{I}) = \phi$

Let $(a_1, \dots, a_n, a_{n+1}) \in k^{n+1}$

Case 1 $\quad (a_1, \dots, a_n) \in V(f_1, \dots, f_s)$

$\Rightarrow f(a_1, \dots, a_n) = 0$ since f vanishes on points in $V(f_1, \dots, f_s)$ (since $f \in I(V(f_1, \dots, f_s))$)

eval at

$\therefore 1 - yf \Rightarrow 1 - a_{n+1} f(a_1, \dots, a_n) = 1 \neq 0 \quad \forall a_{n+1} \in k.$

$\therefore \; (a_1, \dots, a_n, a_{n+1}) \notin V(\tilde{I})$

Case 2: $\quad (a_1, \dots, a_n) \notin V(f_1, \dots, f_s)$

$\Rightarrow$ for at least one f_i

$$f_i(a_1, \dots, a_n) \neq 0$$

$$\Rightarrow f_i(a_1, \dots, a_n, a_{n+1}) \neq 0$$

$$\Rightarrow (a_1, \dots, a_n, a_{n+1}) \notin V(\tilde{I})$$

Putting together case 1, 2:

$$(a_1, \dots, a_{n+1}) \notin V(\tilde{I}) \quad \forall \; (a_1, \dots, a_{n+1}) \in k^{n+1}.$$

$V(\hat{I}) = \phi$

– By weak Nullstellensatz

$\Rightarrow \quad 1 \in \hat{I}$

$\therefore \quad 1 = \sum p_i(x_1,\dots,x_n,y) f_i + q(x_1,\dots,x_n,y)(1-yf)$

Sub $y = \frac{1}{f}$

$$1 = \sum p_i(x_1,\dots,x_n,\tfrac{1}{f}) f_i + \cancel{q(x_1,\dots,x_n,\tfrac{1}{f})\underbrace{(1-\tfrac{1}{f}\cdot f)}_{=0}}$$

Set $m =$ max power of y - variable in p_i

then $f^m = \sum A_i f_i$
$\uparrow$ for some $A_i \in K[x_1,\dots,x_n]$

<u>Lemma</u>: Let V be a variety. If $f^m \in I(V)$
$\Rightarrow f \in I(V)$

<u>def</u>: An ideal I is <u>radical</u> if $f^m \in I$
for some integer $m \geq 1 \Rightarrow f \in I$.

<u>Cor</u> $I(V)$ is a radical ideal

<u>Def</u>: Let $I \subseteq K[x_1,\dots,x_n]$ be an ideal. The <u>radical</u> of I is

$$\mathrm{rad}(I) = \sqrt{I} = \{ f \mid f^m \in I \text{ for some } m \geq 1 \}$$

• Note $I \subseteq \sqrt{I}$

• I radical iff $I = \mathrm{rad}(I)$

Lemma: If I is an ideal then $\sqrt{I}$ is an ideal in $k[x_1, \dots, x_n]$, $I \subseteq \sqrt{I}$, and $\sqrt{I}$ is radical.

Thm (Strong Nullstellen Satz). Let k be an alg. closed field. If J is an ideal in $k[x_1, \dots, x_n]$ then

$$I(V(J)) = \sqrt{J}.$$

Proof: $\sqrt{J} \subseteq I(V(J))$

Since $f \in \sqrt{J} \Rightarrow f^m \in J$

$\therefore f$ vanishes on $V(J)$

$f \in I(V(J))$

If $f \in I(V(J)) \Rightarrow f$ vanishes on $V(J)$

By Nullstellen Satz $\exists\, m \geq 1$ s.t. $f^m \in J$

$\Rightarrow f \in \sqrt{J} \quad \therefore I(V(J)) \subseteq \sqrt{J}$

$\blacksquare$

Thm (The Ideal-Variety Correspondence)

Let k be an arbitrary field.

(i) The maps

$$\text{affine varieties} \xrightarrow[\text{take ideal}]{I} \text{ideals}$$

$$\text{ideals} \xrightarrow[\text{take var.}]{V} \text{affine varieties}$$

are inclusion reversing $(I_2 \subseteq I_1 \iff V(I_1) \supseteq V(I_2))$

(ii) For any variety V

$$V(I(V)) = V$$

So I is always 1-1 and

$$V(\sqrt{J}) = V(J) \quad \text{for any ideal } J$$

(iii) If k alg. closed, we restrict to radical ideals,

The maps

$$\text{affine varieties} \xrightarrow{I \text{ take ideal}} \text{ideals}$$

$$\text{ideals} \xrightarrow{V \text{ take var.}} \text{affine varieties}$$

are inclusion-reversing, bijective, and are inverses of each other.

Proof: (i) done

(ii) $V = V(f_1, \dots, f_s)$

$V \subseteq V(I(V))$ Since $f \in I(V)$ vanishes on V

$V(I(V)) \subseteq V$ Since $f_1, \dots, f_s \in I(V)$

$\Rightarrow (f_1, \dots, f_s) \subset I(V)$

taking variety gives

$$V(I(V)) \subseteq V(f_1, \dots, f_s).$$

(iii) by Nullstellensatz $I(V(J)) = \sqrt{J}$, so if J is radical $\Rightarrow \sqrt{J} = J$. ∎