Thm Every ideal $I \subseteq k[x_1, \dots, x_n]$ can be written as a finite intersection of primary ideals

Proof: call an ideal I <u>irreducible</u> if $I = I_1 \cap I_2$ $\Rightarrow$ either $I = I_1$ or $I = I_2$. Every ideal is the intersection of finitely many irreducibles

[Since: we could construct non-stabilizing accending chain $I \subseteq I_1 \subseteq I_2 \subseteq \cdots$
$I = I_1 \cap I_2 \cap \cdots$
otherwise]

by ACC.

Now show an irreducible ideal is primary

Suppose I irr., $fg \in I$, $f \notin I$

$$I : g^\infty = I : g^N \quad \text{for some } N \text{ (large)}$$

$$(I + (g^N)) \cap (I + (f)) = I$$

(check: ideal is for this N, $fg^N \in I$)

Since I is irreducible $\Rightarrow$ $I = I + (g^N)$

or $I = I + (f)$

Not this since $f \notin I$

$\Rightarrow$ $I = I + (g^N)$ $\Rightarrow$ $g^N \in I$ $\Rightarrow$ I is primary

$\blacksquare$

Def A <u>primary decomposition</u> of an Ideal I is

$$I = \bigcap_{i=1}^{r} Q_i$$

$\uparrow$ primary ideals

It is minimal (or irredundant) if $\sqrt{Q_i}$ are distinct and $Q_i \not\supseteq \bigcap_{j \neq i} Q_j$

Lemma I, J $\swarrow$ in $k[x_1, \dots, x_n]$ primary, $\sqrt{I} = \sqrt{J}$ $\Rightarrow$ $I \cap J$ is primary

Thm (Lasker-Noether decomp. thm)

Every ideal $I \subseteq k[x_1, \dots, x_n]$ has a minimal primary decomposition

Proof: By Thm above $\exists\, Q_i$ s.t. $I = \bigcap_{i=1}^{r} Q_i$

Suppose $\sqrt{Q_i} = \sqrt{Q_j}$ for some $i \neq j$

$\Rightarrow$ $Q = Q_i \cap Q_j$ is primary $\therefore$ replace Q_i, Q_j by Q

Suppose $Q_i \supseteq \bigcap_{i \neq j} Q_j$ $\Rightarrow$ throw away Q_i

this gives a minimal decomp. $\blacksquare$

Lemma) I primary $\sqrt{I} = P$, $f \in k[x_1,\dots,x_n]$

then

- $f \in I \Rightarrow I:f = (1) = k[x_1,\dots,x_n]$
- $f \notin I \Rightarrow I:f$ is P-primary ideal
- $f \notin P \Rightarrow I:f = I$

Thm) Let $I = \bigcap_{i=1}^{r} Q_i$ be a minimal primary decomp. and $P_i = \sqrt{Q_i}$. Then P_i are precisely the proper prime ideals in

$$\{\sqrt{I:f} \mid f \in k[x_1,\dots,x_n]\}$$

Closure Thm

Thm (Closure Thm part 2) k alg. closed $V = V(I) \subseteq k^n$. There exists an affine variety $W \subsetneq V(I)$ s.t

$$V(I_\ell) \setminus W \subseteq \pi_\ell(V) \text{ and } \overline{V(I_\ell) \setminus W} = V(I_\ell)$$

Notation: $k[x_1,\dots,x_\ell, y_{\ell+1},\dots,y_n] = k[x,y]$

Fix an Elimination order i.e. $x^\alpha > x^\beta \Rightarrow x^\alpha > x^\beta y^\gamma$

i.e. Lex with $x_1 > \dots x_\ell > y_{\ell+1} > \dots > y_n$ $\forall \gamma$

$$A \setminus B = A \setminus (A \cap B)$$

Theorem 2. *Fix a field k. Let $I \subseteq k[\mathbf{x}, \mathbf{y}]$ be an ideal and let $G = \{g_1, \dots, g_t\}$ be a Gröbner basis for I with respect to a monomial order as above. For $1 \le i \le t$ with $g_i \notin k[\mathbf{y}]$, write g_i in the form*

$$g_i = c_i(\mathbf{y})\mathbf{x}^{\alpha_i} + \textit{terms} < \mathbf{x}^{\alpha_i}. \tag{1}$$

Finally, assume that $\mathbf{b} = (a_{l+1}, \dots, a_n) \in \mathbf{V}(I_l) \subseteq k^{n-l}$ is a partial solution such that $c_i(\mathbf{b}) \neq 0$ for all $g_i \notin k[\mathbf{y}]$. Then:

(i) *The set*

$$\overline{G} = \{g_i(\mathbf{x}, \mathbf{b}) \mid g_i \notin k[\mathbf{y}]\} \subseteq k[\mathbf{x}]$$

is a Gröbner basis of the ideal $\{f(\mathbf{x}, \mathbf{b}) \mid f \in I\}$.

(ii) *If k is algebraically closed, then there exists $\mathbf{a} = (a_1, \dots, a_l) \in k^l$ such that $(\mathbf{a}, \mathbf{b}) \in V = \mathbf{V}(I)$.*

Proof: (i) Skip

(ii) Note that $\overline{g_i}$ is non-constant $\forall i$ (since $g_i \notin k[y]$)

$\Rightarrow 1 \notin \overline{I}$ ← I evaluated at b $\therefore \overline{I} \subsetneq k[x]$ ← proper

$\therefore$ by Nullstellensatz $V(\overline{I})$ is non-empty

$\therefore \exists\, a \in k^l$

s.t. $a \in V(\overline{I}) = V(\overline{G})$

$\Rightarrow \overline{g_i}(a) = 0 \ \forall\, \overline{g_i} \in \overline{G}$

$\therefore g_i(a, b) = 0 \ \forall i$ since if $g_i \in G \setminus k[y]$ and b is partial sol then $g_i(a,b) = 0$

if $g_i \in k[y] \Rightarrow g_i(b) = g_i(a,b) = 0$

$\therefore (a,b) \in V = V(I)$ ∎