Prop $I \subseteq k[x_1, \dots, x_n] \leftrightarrow I(V)$ for $V = \{(a_1, \dots, a_n)\} \subseteq k^n$

$I = (x_1 - a_1, \dots, x_n - a_n)$

is a maximal ideal.

Proof: Suppose $J \supsetneq I \Rightarrow \exists f \in J$ s.t $f \notin I$

by div. alg.

$$f = A_1(x_1 - a_1) + \dots + A_n(x_n - a_n) + \underset{\uparrow\ b \in k}{b}$$

$f \notin I$

$\therefore b \neq 0 \quad f \in J$

$\therefore b \in J$

$\Rightarrow 1 \in J \Rightarrow J = k[x_1, \dots, x_n]$

I is maximal $\blacksquare$

$V(x_1 - a_1, \dots, x_n - a_n) = \{(a_1, \dots, a_n)\}$

$\therefore$ Every point in k^n gives a maximal ideal

Converse false over k - not alg. closed

Thm If k alg. closed then every maximal ideal of $k[x_1, \dots, x_n]$ has the form $(x_1 - a_1, \dots, x_n - a_n)$ $a_i \in k$.

Proof: $I \subseteq k[x_1, \dots, x_n]$ maximal. $I \neq k[x_1, \dots, x_n]$

$\therefore V(I) \neq \emptyset$ by weak Nullstellensatz

$\therefore \quad (a_1, \dots, a_n) \in V(I) \quad$ for some $a_i \in k$

$f \in I \implies f \in I(\{(a_1, \dots, a_n)\})$
$= (x_1 - a_1, \dots, x_n - a_n)$

$\therefore \quad I \subseteq (x_1 - a_1, \dots, x_n - a_n) \subsetneq k[x_1, \dots, x_n]$

(since maximal)

$\implies \quad I = (x_1 - a_1, \dots, x_n - a_n)$ ∎

cor.) k alg. closed. There is a 1-1 correspondence between points of k^n and maximal ideals of $k[x_1, \dots, x_n]$.

Prop) V is irreducible iff for every $W \subsetneq V$, $V \setminus W$ is Zariski dense in V, i.e. $\overline{V \setminus W} = V$.

Proof: Take V irr. $W \subsetneq V$
↑ a variety

$$V = W \cup \overline{V \setminus W}$$

V is irr. and $V \neq W$ by def of irriducible

$$V = \overline{V \setminus W}$$

Now take $V = V_1 \cup V_2$. If $V_1 \subsetneq V$, $\overline{V \setminus V_1} = V$ by assumtion

But $V \setminus V_1 \subseteq V_2 \implies \overline{V \setminus V_1} \subseteq V_2 \implies V \subseteq V_2$

$\Rightarrow \quad V = V_2 \quad \therefore \quad V$ is irreducible $\blacksquare$

$S \subseteq V$

Zariski dense if $\overline{S} = V$

$S \subseteq W \subsetneq V$

Decompositions into irr.

Prop 1 (Descending chain con.). DCC

Any decending chain of varieties

$$V_1 \supseteq V_2 \supseteq \cdots \quad \text{in } k^n$$

must stabalize, i.e. $\exists\, N > 0$ s.t. $V_N = V_{N+1} = \cdots$.

Proof) ACC

$$I(V_1) \subseteq I(V_2) \subseteq \cdots$$

Thm) Let $V \subseteq k^n$ be a variety. we can write

$$V = V_1 \cup \cdots \cup V_m$$

where each V_i is an irreducible variety.

Proof:

Assume V can't be written as a finite union of irribuble

$\Rightarrow$ V is not irreducible

$\Rightarrow$ $V = V_1 \cup \tilde{V}_1$ $\quad V \neq V_1$ and $V \neq \tilde{V}_1$

$\Rightarrow$ $V_1 = V_2 \cup \tilde{V}_2$ $\quad V_1 \neq V_2, \; V_1 \neq \tilde{V}_2$

V_2 not a finite union of irr...

$\Rightarrow$ we have

$$V \supseteq V_1 \supseteq V_2 \supseteq \cdots\cdots$$

$V \neq V_1 \neq V_2 \neq \cdots\cdot$ This contradicts the D.C.C. $\boxtimes$

$\therefore$

$$V(xz, yz) = V(x,y) \cup V(z)$$

is this unique

Def) Let $V \subseteq k^n$ a variety A decomp.

$$V = V_1 \cup \cdots \cup V_m$$

is a minimal decomp. if $V_i \not\subseteq V_j$ for $i \neq j$

Call V_i the irreducible components of V.

Thm) A variety $V \subseteq k^n$ has a minimal decomp.

$$V = V_1 \cup \cdots \cup V_m \leftarrow \text{irreducibles}$$

and this decomp is unique

Proof : we know $V = V_1 \cup \cdots \cup V_{\hat{m}}$ exists for V_i irreducible. If $V_i \subseteq V_j$ $i \neq j$ remove V_i until no such inclusion exists. This gives a minimal decomp.

$$V = V_1 \cup \cdots \cup V_m$$

Uniquness :

Suppose $V = \tilde{V}_1 \cup \cdots \cup \tilde{V}_\ell$ is also a minimal decomp

$$V_i = V_i \cap V = V_i \cap (\tilde{V}_1 \cup \cdots \cup \tilde{V}_\ell)$$
$$= (V_i \cap \tilde{V}_1) \cup \cdots \cup (V_i \cap \tilde{V}_\ell)$$

V_i is irreducible $\therefore$ $V_i = V_i \cap \tilde{V}_j$ for some j

$$V_i \subseteq \tilde{V}_j$$

Same procedure gives $\tilde{V}_j \subseteq V_K$

$\therefore$ $V_i \subseteq \tilde{V}_j \subseteq V_K$ $\therefore$ $V_i = \tilde{V}_j = V_K$ by minimality.

$\therefore$ every V_i appears in $V = \tilde{V}_1 \cup \cdots \cup \tilde{V}_\ell$

$\therefore$ $m \leq \ell$, but same procedure gives $\ell \leq m$

$\therefore$ $m = \ell$ and one decomp. is a permutation of the other. ∎

Uniqueness shows def of irreducible compents are well-defined ... finite is important

Prop) V, W varieties, $W \subsetneq V$, $V \setminus W$ is Zariski dense in $V = \overline{V \setminus W}$ iff W contains no irreducible components of V.

Proof: Suppose $V = V_1 \cup \cdots \cup V_m$ (minimal) and that V_i is not contained in W $\forall i$

not in W

$$V_i \cap W \subsetneq V_i$$

V_i is irreducible $\therefore$ $\overline{V_i \setminus (V_i \cap W)} = V_i$

$$\therefore \overline{V \setminus W} = \overline{(V_1 \cup \cdots \cup V_m) \setminus W}$$

$$= \overline{V_1 \setminus (V_1 \cap W)} \cup \cdots \cup \overline{V_m \setminus (V_m \cap W)}$$

$$= V_1 \cup \cdots \cup V_m = V$$

Other direction an excise ∎

Thm) If k is alg. closed then every radical ideal I in $k[x_1, \ldots, x_n]$ can be written uniquely as

$$I = P_1 \cap \cdots \cap P_m$$

where P_i is prime and $P_i \not\subset P_j$ for $i \neq j$.

(minimal prime decomp.)

Proof:) Theorem + ideal-variety cor.

Thm] If k alg. closed $I \subsetneq k[x_1, \dots, x_n]$ is radical with minimal decomp

$$I = \bigcap_{i=1}^{m} P_i$$

then the P_i's are preciseley the proper prime ideals that occur in the set $\{ I : f \mid f \in k[x_1, \dots, x_n] \}$

proof:] I is proper and decomp is minimal
$\Rightarrow$ P_i is propen $\forall i$ (otherwise $P_i \subseteq P_j$ for some i, j)

$f \in k[x_1, \dots, x_n]$

#17 4.4

$$I : f = \left(\bigcap_{i=1}^{m} P_i \right) : f = \bigcap_{i=1}^{m} (P_i : f)$$

Since P_i is prime

if $f \in P_i \Rightarrow P_i : f = (1)$

if $f \notin P \Rightarrow P_i f = P$

Suppose $I : f$ is proper, prime (By #4, 4.5)

$\Rightarrow I : f = P_i : f$ for some i

$\Rightarrow I : f = P_i$

Show every P_i can be obtained in this way

Fix i. Pick $f \in \left(\bigcap_{i \neq j}^{m} P_j \right) \setminus P_i$

$\Rightarrow$ $P_i : f = P_i$, $P_j : f = (1)$ $i \neq j$

$I : f = P_i$

■

- Note last two thm's hold over <u>any</u> field k.

<u>Primary Decompositions</u>

- decomposed radical ideals into primes
- What about arbitrary ideals.

<u>Def)</u> An ideal I is <u>primary</u> (in $k[x_1, \dots, x_n]$) if $f \cdot g \in I$
$\Rightarrow$ either $f \in I$ or $g^m \in I$ for some m.

<u>Lemma)</u> If I is primary $\Rightarrow$ $\sqrt{I}$ is prime and $\sqrt{I}$ is the smallest prime ideal containing I.

<u>Def)</u> If I is primary and $\sqrt{I} = P$ (<u>associated prime</u>) then we say I is <u>P-primary</u>.

Thm | Every ideal $I \subseteq k[x_1, \dots, x_n]$ can be written as a finite intersection of primary ideals

Proof: call an ideal I irreducible if $I = I_1 \cap I_2$ $\Rightarrow$ either $I = I_1$ or $I = I_2$. Every ideal is the intersection of finitely many irreducibles

[since: we could construct non-stabalizing accending chain $I \subseteq I_1 \subseteq I_2 \subseteq \cdots$
$I = I_1 \cap I_2 \cap \cdots$
otherwise]

by ACC.

Now show