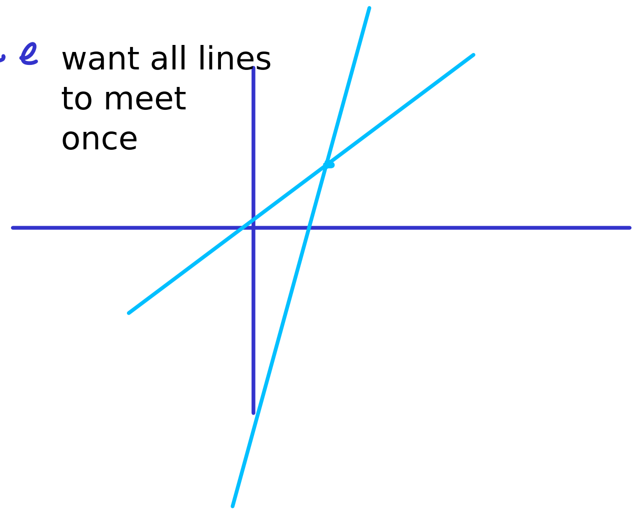
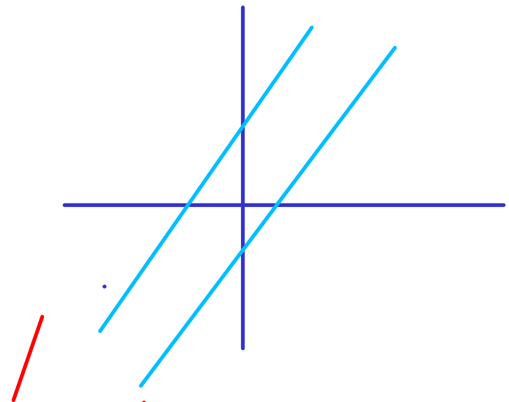
Projective Space



we want all lines to meet once



Parallel lines to meet at a "point at ∞"

Consider points in k^{n+1} define an eq relation on <u>non-zero</u> points of k^{n+1}

$$(x_0,\dots,x_n) \sim (\lambda x_0, \dots, \lambda x_n) \quad \forall \lambda \neq 0 \in k$$

Def Projective space of dim n over a field k $\mathbb{P}^n(k)$ or just $\mathbb{P}^n$ is the set of equivalence classes of $\sim$ on $k^{n+1}/\{0\}$ i.e.

$$\mathbb{P}^n = (k^{n+1}\setminus\{0\})/\sim$$

Points $p \in \mathbb{P}^n$ written as $p=(a_0:\dots:a_n)$ called <u>homogeneous coordinates</u> of p

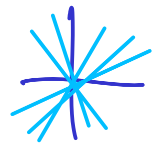
$$(\tilde{x}_0:\dots:\tilde{x}_n) = (x_0:\dots:x_n)$$

$$\Updownarrow$$

$(\hat{x}_0, \ldots, \hat{x}_n) = \lambda(x_0, \ldots, x_n)$ for some $\lambda \in k \setminus \{0\}$

$\mathbb{P}^n \cong \{\text{lines through the origin in } k^{n+1}\}$

each correspondes to a line.

 $\cong \mathbb{P}^1$

$\boxed{\text{Prop}}$ Let $U_0 = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n \mid x_0 \neq 0\}$

$$\phi : k^n \longrightarrow U_0 \subseteq \mathbb{P}^n$$
$$: (a_1, \ldots, a_n) \longmapsto (1 : a_1 : \cdots : a_n)$$

Then this is a bijection.

<u>Proof:</u> Since $\phi(a_1, \ldots, a_n) = (1 : a_1 : \cdots : a_n)$

we have a 1-1 map $k^n \to U_0$

Define an inverse map

Given $p = (x_0 : \cdots : x_n) \in U_0$, since $x_0 \neq 0$

$$= \left(1 : \frac{x_1}{x_0} : \cdots : \frac{x_n}{x_0}\right)$$ mult by $\lambda = \frac{1}{x_0}$

Let $\theta : p \longmapsto \left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$

$\phi(\theta(p)) = p$ $\blacksquare$

Note from this we have

$$\mathbb{P}^n = \underbrace{U_0}^{\simeq k^n} \cup H$$

hyperplane at infinity $x_0 = 0$

$$H = \{ p \in \mathbb{P}^n \mid p = (0 : x_1 : \cdots : x_n) \}$$

$\updownarrow$ 1-1

$\mathbb{P}^{n-1}$

$$\mathbb{P}^n = k^n \cup \mathbb{P}^{n-1}$$ ← point at infinity

point p gives a line $L \subseteq k^n$ through origion

$\therefore$ p represents asymptotic direction of all lines in k^n parallel to L

this is the point in $\mathbb{P}^n$ where all lines parallel to L meet

Corollary 3. *For each $i = 0, \ldots n$, let*

$$U_i = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n(k) \mid x_i \neq 0\}.$$

$U_i \simeq k^n$

(i) *The points of each U_i are in one-to-one correspondence with the points of k^n.*
(ii) *The complement $\mathbb{P}^n(k) \setminus U_i$ may be identified with $\mathbb{P}^{n-1}(k)$.* $\Rightarrow$ $\mathbb{P}^n \simeq k^n \cup \mathbb{P}^{n-1}$
(iii) *We have $\mathbb{P}^n(k) = \bigcup_{i=0}^n U_i$.* – U_i's cover $\mathbb{P}^n = U_i \cup \mathbb{P}^{n-1}$

Q: How do we define projective varieties

A: Homogeneous eqn's

Ex] Does $V(\overset{f}{\overbrace{x_1 - x_2^2}})$ in $\mathbb{P}^2$ make sense?

A: NO! i.e $p = (1:4:2) = (2:8:4) \in \mathbb{P}^2$.

$4 - 2^2 = 0$ $\quad\quad$ $8 - 4^2 = -8 \neq 0$

$f(p) = 0$ $\quad\quad$ $f(p) \neq 0$

Def] $f \in k[x_0, \dots, x_n]$ is homogeneous polynomial of total degree d if every monomial in f has degree d.

Ex] $x_1 - x_2^2$ $\quad\quad$ $x_0 x_1 - x_2^2$

Not homogeneous $\quad\quad$ is homogeneous

Prop] Let $f \in k[x_0, \dots, x_n]$ be homogeneous. Then

$$V(f) = \{ p \in \mathbb{P}^n \mid f(p) = 0 \} \subseteq \mathbb{P}^n$$

is well defined $\Leftrightarrow$ [f vanishes on one homo-coord

$\Updownarrow$

f vanishes on all hom. coord]

Proof: Let $p = (a_0 : \cdots : a_n) = (\lambda a_0 : \cdots : \lambda a_n) \in \mathbb{P}^n$

$(\lambda \neq 0 \in k)$

and assume $f(a_0, \ldots, a_n) = 0$, f is homogeneous of degree d.

$$f = \sum_{\alpha_0 + \cdots + \alpha_n = d} c_\alpha \, x_0^{\alpha_0} \cdots x_n^{\alpha_n}$$

$$\lambda^{\alpha_0} \cdot \lambda^{\alpha_1} \cdots \lambda^{\alpha_n} = \lambda^{\alpha_0 + \cdots + \alpha_n}$$

$$f(\lambda a_0, \ldots, \lambda a_n) = \sum_{\alpha_0 + \cdots + \alpha_n = d} c_\alpha \underbrace{(\lambda a_0)^{\alpha_0} \cdots (\lambda a_n)^{\alpha_n}}$$

$$= \sum_{\alpha_0 + \cdots + \alpha_n = d} c_\alpha \cdot \lambda^d (a_0)^{\alpha_0} \cdots (a_n)^{\alpha_n}$$

$$= \lambda^d \sum_{\alpha_0 + \cdots + \alpha_n = d} c_\alpha (a_0)^{\alpha_0} \cdots (a_n)^{\alpha_n}$$

$$= \lambda^d f(a_0, \ldots, a_n) = 0$$

$\blacksquare$