Ex) $I = (x^2 + y - 1,\ xy - 2y^2 + 2y) \subseteq k[x,y]$

lex GB gives $(LT(I)) = (x^2, xy, y^3)$

A vector space basis for $k[x,y]/I$

is $\{1, x, y, y^2\}$

$$\underset{\in k[x,y]/I}{f \cdot g} = (3y^2 + x)\cdot(x - y) = 3xy^2 + x^2 - 3y^2 - xy$$

Not st. rep

$$\overline{f \cdot g}^G = -\frac{11}{4}y^2 - \frac{5}{4}y + 1$$

Proposition 5. *Let I be an ideal in $k[x_1, \dots, x_n]$ and let G be a Gröbner basis of I with respect to any monomial order. For each $[f] \in k[x_1, \dots, x_n]/I$, we get the standard representative $\overline{f} = \overline{f}^G$ in $S = \mathrm{Span}(x^\alpha \mid x^\alpha \notin \langle \mathrm{LT}(I) \rangle)$. Then:*

(i) *$[f] + [g]$ is represented by $\overline{f} + \overline{g}$.*

(ii) *$[f] \cdot [g]$ is represented by $\overline{\overline{f} \cdot \overline{g}} \in S$.*

$k[V]$

$I = I(V)$

Do your ops. in $k[V]$ and reduce w.r.t. a GB of I.

Thm) (Finiteness Thm) Fix a monomial ordering $I \subseteq k[x_1, \dots, x_n]$ an ideal.

(i) For each i $1 \le i \le n$ $\exists\, m_i \ge 0$ s.t $x_i^{m_i} \in \langle LT(I) \rangle$

(ii) G a GB for I

$x^{m_i} = LM(g)$ for some $g \in G$

this is same

(iii) $\{ x^\alpha \mid x^\alpha \notin \langle LT(I) \rangle \}$ is finite

(iv) $\dim_k \left(k[x_1, \dots, x_n]/I \right) < \infty$

(v) If k alg closed $V(I) \subseteq k^n$ is a finite set.

(i) - (iv) are equivalent $\forall k$. (i) throug (v) are equivilent for k. alg closed

Proof:

(iii) $\Leftrightarrow$ (iv) From Prop.

(i) $\Leftrightarrow$ (ii) ch. 2.

(i) $\Leftrightarrow$ (iii)

Show (i) $\Rightarrow$ (iii)

If $x_i^{m_i} \in \langle LT(I) \rangle$ for each i

then the $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for which some $\alpha_i \ge m_i$ are all in $\langle LT(I) \rangle$

$\therefore$ The monomials in complement of $(LT(I))$ must have $0 \le d_i \le m_i - 1$ $\forall i$

$\therefore$ number of monomials in complement is at $m_1 \cdots m_n$

(iii) $\Rightarrow$ (i) Suppose complement has $N < \infty$ monomials

$\Rightarrow$ for each i at least 1 of $1, x_i, \dots, x_i^N$ is in $(LT(I))$

(iv) $\Rightarrow$ (v)

$\dim_K (K[x_1, \dots, x_n] / I) < \infty$

Show $\exists$ finitely many distinct ith coords of points of V.

Fix an arbitrary i.

Finite dim.

Consider $[x_i^j] \in K[x_1, \dots, x_n]/I$ $j = 0, 1, \dots$

family of coord powers.

$\therefore$ $[x_i^j]$ must be lin dependent

$\exists$ c_j (not all zero) and m s.t.

$$[0] = \sum_{j=0}^{m} c_j [x_i^j] = \left[\sum c_j x_i^j \right]$$

$\therefore \quad \sum' c_j x_i^j \in I$

$$\Rightarrow \left(\sum' c_j^{(1)} x_1^j, \ldots, \sum' c_j^{(n)} x_n^j \right) \subseteq I$$

a 1-var poly can have only a finite number of solutions

$$\Rightarrow V(I) \subseteq V\left(\sum' c_j^{(1)} x_1^j, \ldots, \sum' c_j^{(n)} x_n^j \right)$$

$\therefore V(I)$ is finite.

Assume k is alg. closed show $(v) \Rightarrow (i)$

$V(I)$ is finite $\Rightarrow$ If $V = \emptyset$ then $1 \in I$ by weak Nullstelentatz $\therefore x_i^0 \in (LT(I)) \ \forall i$

If V is non-empty, fix i, let $a_1, \ldots, a_m \in k$ be the distinct i^{th} of points in V

Consider $f(x_i) = \prod_{j=1}^{m} (x_i - a_j)$

$$\Rightarrow f \in I(V)$$

By Nullstel. $\Rightarrow f^N \in I$

$$\Rightarrow LM(f^N) = x_i^{mN} \in (LT(I)) \quad \blacksquare$$

Prop Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal s.t. $\dim_k (k[x_1, \dots, x_n]/I) < \infty$ (and $\Leftrightarrow$ $x_i^{m_i} \in (LT(I))$ for some m_i $\forall i$)

(i) # points in $V \le \dim_k (k[x_1, \dots, x_n]/I)$

(ii) # points in $V \le m_1 \cdots m_n$

(iii) If I radical, k alg. closed

$$\dim_k (k[x_1, \dots, x_n]/I) = \# \text{ points in } V.$$

Proof (sketch)

- Show given distinct points $p_1, \dots, p_m \in k^n$

$\exists$ $f_1, \dots, f_m$ s.t. $f_i(p_i) = 1$ and $f_i(p_j) = 0$ $\forall i \neq j$

V is finite, $V = \{p_1, \dots, p_m\}$ distinct

prove that $[f_1], \dots, [f_m] \in k[x_1, \dots, x_n]/I$ are lin. independent

[Suppose $a_i \in k$ $\sum a_i [f_i] = [0]$

$\Rightarrow$ $g = \sum a_i f_i \in I$

$\therefore$ $0 = g(p_i) = a_i$]

$$\underbrace{m}_{\# \text{ points in } V} \le \dim_k (k[x_1, \dots, x_n]/I)$$

(vii) Taking k alg closed, I radical

Show $[f_1], \dots, [f_m]$ is a basis for $k[x_1, \dots, x_n]/I$ $k[V]$ "

Let $[g] \in k[V]$ be arbitrary

$\exists\, a_i \in k$ s.t

$$h = g - \sum \underbrace{a_i}_{\in k} f_i \in I(V) \quad \text{(check this)}$$

$a_i = g(P_i)$

by Null. $h \in I$ (since I is radical)

$$\Rightarrow [g] = \sum a_i [f_i]$$

Recall coordinate ring of $V \subseteq k^n$

is $k[V] \simeq k[x_1, \dots, x_n]/I(V)$

Def For any ideal $J = (\phi_1, \dots, \phi_s) \subseteq k[V]$ define

- $V_V(J) = \{(a_1, \dots, a_n) \in V \mid \phi(a_1, \dots, a_n) = 0 \;\forall \phi \in J\}$

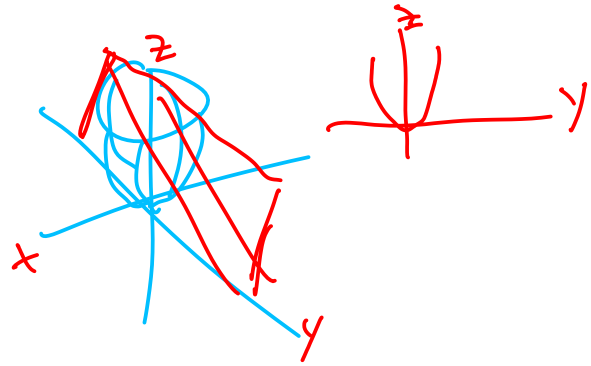
  subvariety of V

- For each subset $W \subseteq V$ define

$$I_V(W) = \{\phi \in k[V] \mid \phi(a_1, \dots, a_n) = 0 \;\forall (a_1, \dots, a_n) \in W\}$$

Ex $V = V(\underbrace{z - x^2 - y^2}_{f=}) \subseteq \mathbb{R}^3$

↑ paraboloid



$J = ([x]) \subseteq \mathbb{R}[V]$

↓ subvariety where $x = 0$ i.e. $[x] = 0$ in $\mathbb{R}[x,y,z]/(f)$

$W = V_V(J) = \{(0, a, a^2) \mid a \in \mathbb{R}\}$

$\|$

$W = V(z - \overset{0}{x^2} - y^2, x)$

Proposition 3. *Let $V \subseteq k^n$ be an affine variety.*

(i) *For each ideal $J \subseteq k[V]$, $W = \mathbf{V}_V(J)$ is an affine variety in k^n contained in V.* ~ it really is a subvariety

(ii) *For each subset $W \subseteq V$, $\mathbf{I}_V(W)$ is an ideal of $k[V]$.*

(iii) *If $J \subseteq k[V]$ is an ideal, then $J \subseteq \sqrt{J} \subseteq \mathbf{I}_V(\mathbf{V}_V(J))$.*

(iv) *If $W \subseteq V$ is a subvariety, then $W = \mathbf{V}_V(\mathbf{I}_V(W))$.*

Proof: (i)

Recall: ideals in $k[V] \overset{1-1}{\longleftrightarrow}$ ideals in $k[x_1, \dots, x_n]$

$J \subseteq k[V]$ an ideal containing $I(V)$

$\tilde{J} = \{f \in k[x_1, \dots, x_n] \mid [f] \in J\} \subseteq k[x_1, \dots, x_n]$

this is ideal corresponding to $J \subseteq k[V]$

$V(\tilde{J}) \subseteq V$ since $I(V) \subseteq \tilde{J}$

but $W = V(\tilde{J}) = V_V(J)$ by def

aut time $[f](p) = 0$ in $k[V]$ $f(p) = 0$ and $(f+g)(p) = 0$

$\forall g \in I(V)$

Prop) $J \subseteq k[V]$ is radical iff corresponding ideal

$$\tilde{J} = \{f \in k[x_1,\dots,x_n] \mid [f] \in J\} \subseteq k[x_1,\dots,x_n]$$

is radical.

Proof: Show J radical $\Rightarrow \tilde{J}$ radical

J is radical. Let $f \in k[x_1,\dots,x_n]$ s.t $f^m \in \tilde{J}$

$$\Rightarrow \underbrace{[f^m]}_{\in J} = [f]^m \in J \Rightarrow [f] \in J$$

$$\Rightarrow f \in \tilde{J}$$

other direction similar $\boxtimes$

Theorem 5. *Let k be an algebraically closed field and let $V \subseteq k^n$ be an affine variety.*

(i) (**The Nullstellensatz in $k[V]$**) *If J is any ideal in $k[V]$, then*

$$\mathbf{I}_V(\mathbf{V}_V(J)) = \sqrt{J} = \{[f] \in k[V] \mid [f]^m \in J\}.$$

(ii) *The correspondences*

$$\left\{\begin{matrix} \text{affine subvarieties} \\ W \subseteq V \end{matrix}\right\} \underset{\mathbf{V}_V}{\overset{\mathbf{I}_V}{\rightleftarrows}} \left\{\begin{matrix} \text{radical ideals} \\ J \subseteq k[V] \end{matrix}\right\}$$

are inclusion-reversing bijections and are inverses of each other.

(iii) *Under the correspondence given in (ii), points of V correspond to maximal ideals of $k[V]$.*

Proof: write $\tilde{J}$, apply affine results.

Def) $V \subseteq k^m$, $W \subseteq k^n$ aff. varieties

$V \cong W$ iff $\exists$ poly mappings $\alpha: V \to W$ and

$\uparrow$ isomorphic

$\beta: W \to V$ s.t $\alpha \circ \beta = id_W$ and $\beta \circ \alpha = id_V$

Thm $V \subseteq k^m$, $W \subseteq k^n$. $W \simeq V$ iff

$k[V] \overset{\Phi}{\simeq} k[W]$ where Φ is the identity on

ring isomorphism

constant functions.