Irr. Varieties and Prime Ideals

Ex) $V(xz, yz) = V(x,y) \cup V(z) \subseteq k^3$

irriducible

Def) $V \subseteq k^n$, a variety, is irreducible if whenever we write $V = V_1 \cup V_2$, V_1, V_2 varieties in k^n

$\Rightarrow$ Either $V_1 = V$, $V_2 = V$.

reducible $\Leftrightarrow$ not irreducible.

Def) I (an ideal) is prime if whenever

$fg \in I \Rightarrow$ Either $f \in I$ or $g \in I$.

Prop) $V \subseteq k^n$ a variety is irreducible if and only if $I(V)$ is a prime ideal.

Proof: • Take V irr. $fg \in I(V)$. Set

$V_1 = V \cap V(f)$, $V_2 = V \cap V(g)$

$\boxed{\text{Ex1} \quad V(f_1, f_2) \cap V(f_1) = V(f_1, f_2)}$

$fg \in I(V) \Rightarrow V = V_1 \cup V_2 = (V \cap (V(f) \cup V(g))) = V \cap V(fg))$

$V \cap V(fg) = V$

V is irr. $\Rightarrow V = V_1$ or $V = V_2$

WLOG $V = V_1 \Rightarrow f \in I(V)$

$\Rightarrow I(V)$ is prime

- Assume $I(V)$ is prime, let $V = V_1 \cup V_2$
  - Suppose $V \neq V_1$, (show $I(V) = I(V_2)$)
  - $I(V) \subseteq I(V_2)$ (since $V_2 \subset V$)
  - $V_1 \subsetneq V \Rightarrow I(V) \subsetneq I(V_1)$

Pick $f \in I(V_1) \setminus I(V)$ and any $g \in I(V_2)$

$\Rightarrow f \cdot g \in I(V)$ Since $V = V_1 \cup V_2$

$I(V)$ is prime $\therefore f \in I(V)$ or $g \in I(V)$

but $f \notin I(V)$

$\therefore g \in I(V)$ $I(V_2) \leq I(V) \Rightarrow I(V_2) = I(V)$

$\Rightarrow V = V_2$ $\therefore V$ is irreducible $\boxtimes$

Every prime ideal is radical

[Ex: $f \cdot f^2 \in I \Rightarrow f \in I$ or $f^2 \in I$]

Corol when k is Alg. closed

$$\text{Prime Ideals} \underset{I}{\overset{V}{\rightleftarrows}} \text{irreducible varieties} \quad 1\text{-}1$$

Ex) twisted cubic (k - infinite) $= \overline{\{(t,t^2,t^3) \mid t\in k\}} = V$

Suppose $f\cdot g \in I(V)$

$$\Rightarrow \quad f(t,t^2,t^3)\,g(t,t^2,t^3)=0$$

either $f(t,t^2,t^3)=0$ or $g(t,t^2,t^3)=0$

$\Rightarrow f\in I(V)$ or $g\in I(V)$

$\therefore I(V)$ is prime

$\therefore$ twisted cubic is irr.

Prop) k infinite $F: k^m \longrightarrow k^n$

$$(t_1,\dots,t_m) \longmapsto (f_1(t_1,\dots,t_m),\dots,f_n(t_1,\dots,t_m))$$

$(t_1,\dots,t_m) = \mathbf{t}$

$V = \overline{F(k^m)}$ is irreducible.

$$\left(I(V) = I(F(k^m)) \right)$$

Proof:

For any $g\in k[x_1,\dots,x_n]$

$$g\circ F = g(f_1(\mathbf{t}),\dots,f_n(\mathbf{t})) \in k[t_1,\dots,t_m]$$

$$I(V) = I(F(k^m)) = \{ g\in k[x_1,\dots,x_n] \mid g\circ F = 0\}$$

Suppose $gh \in I(V)$

$$0 = gh\circ F = gh(f_1,\dots,f_n) = g(f_1,\dots,f_n)\,h(f_1,\dots,f_n)$$

$$= g \circ F \cdot h \circ F = 0$$

$\therefore$ Either $g \circ F = 0$ or $h \circ F = 0$

$\Rightarrow g \in I(V)$ or $h \in I(V)$

$\Rightarrow V$ irreducible

Prop] k infinite. $W = V(g_1 \cdots g_n)$, $f_1, \dots, f_n, g_1, \dots, g_n \in k[t_1, \dots, t_m]$

$$F : k^m \setminus W \longrightarrow k^n$$

$$(t_1, \dots, t_m) \longmapsto \left(\frac{f_1(t)}{g_1(t)}, \dots, \frac{f_n(t)}{g_n(t)} \right)$$ then

$V = \overline{F(k^m \setminus W)}$ is irreducible.

$\Updownarrow$

$I(\overline{F(k^m \setminus W)})$ is prime.

Proof: $h \in k[x_1, \dots, x_n]$

$$g_1(t) \cdots g_n(t) \neq 0 \quad \forall t \in k^m \setminus W$$

$$\therefore (g_1 \cdots g_n)^N \cdot (h \circ F) = 0 \quad \forall t \in k^m \setminus W$$

$$\Updownarrow$$

$$h \circ F = 0$$

If $N = \underset{\text{total degree}}{\deg}(h)$ then $(g_1 \cdots g_n)^N h \circ F \in k[t_1, \dots, t_m]$

$\therefore h \in I(V)$ iff $(g_1 \cdots g_n)^N (h \circ F) = 0 \in k[t_1, \dots, t_m]$

$p, q \in k[x_1, \ldots, x_n]$

$p \cdot q \in I(V) \qquad \deg(p) = M, \quad \deg(q) = N$

$\therefore \quad (g_1 \cdots g_n)^{N+M} (p \cdot q(F(t))) = 0$

$(g_1 \cdots g_n)^{M} p(F(t)) \cdot (g_1 \cdots g_n)^{N} q \cdot (F(t)) = 0$

$\therefore \quad p \in I(V) \quad \text{or} \quad q \in I(V)$

$\blacksquare$