To Find W in Closure thm:

Algorithm FindW:

Input: $I \subseteq K[x, y]$ ($x_1, \dots, x_\ell, y_{\ell+1}, \dots, y_n$) with $V = V(I)$

Output: $W \subseteq V(I_\ell)$ with $V(I_\ell) \setminus W \subseteq \pi_\ell(V)$

and

$$\overline{V(I_\ell) \setminus W} = V(I_\ell)$$

DO: G = reduced GB w.r.t an elim order (for x)

$c_i = c_i(y)$ from $g_i = c_i(y) x^{\alpha_i}$ + smaller terms.
$g_i \in G \setminus K[y]$

$I_\ell = I \cap K[y] = G \cap K[y]$

$$J = \prod_{g_i \in G \setminus K[y]} c_i(y)$$

If $\overline{V(I_\ell) \setminus V(J)} = V(I_\ell)$

return $W = V(I_\ell) \cap V(J)$

Else Pick $g_i \in G \setminus K[y]$, $I \neq I : c_i^\infty$

return $W = \text{FindW}(I + (c_i)) \cup \text{FindW}(I : c_i^\infty)$

Thm Let K be alg. closed. V an affine variety.
Then $\exists$ affine varieties $Z_i \subseteq W_i \subseteq K^n$ for $1 \le i \le p$
s.t. $\pi_\ell(V) = \bigcup_{i=1}^{p} (W_i \setminus Z_i)$ ← This is called a constructible set.

Poly. Mappings

Def $V \subseteq k^m$, $W \subseteq k^n$ varieties

$\phi: V \longrightarrow W$ is a polynomial (or regular) mapping if $\exists$ polynomials $f_1, \dots, f_n \in k[x_1, \dots, x_m]$ s.t.

$$\phi(a_1, \dots, a_m) = (f_1(a_1, \dots, a_m), \dots, f_n(a_1, \dots, a_m)) \in W$$

$\forall (a_1, \dots, a_m) \in V$

$\uparrow$ satisfy defining eqs of W $\forall (a_1, \dots, a_n) \in V$

We say $(f_1, \dots, f_n) \in k[x_1, \dots, x_m]^n$ represents ϕ.
f_i are the components of this rep.

Ex $V = V(y - x^2, z - x^3) \subseteq k^3$ ← twisted cubic — Parametrically (a, a^2, a^3)

$W = (y^3 - z^2) \subseteq k^2$

the Projection $\pi_1 : k^3 \to k^2$
$(x, y, z) \longmapsto (y, z)$

is represented by (y, z) and gives a poly. mapping $\pi_1 : V \to W$

since $\pi_1(V) = \{(a^2, a^3) \mid a \in k\}$

$y^3 - z^2$ vanishes everywhere in $\pi_1(V)$ since

$$(a^2)^3 - (a^3)^2 = 0$$

We want to understand all maps ($V \subseteq k^m$)

$$\phi : V \longrightarrow k^n$$

Sufficient to look at components i.e. to look at

$$\phi : V \longrightarrow k$$

Suppose $\phi : V \to k$
$a \mapsto f(a)$

f is a rep. of ϕ but if $g \in I(V)$ $f+g$ is also a rep (since $g(a) = 0$ $\forall a \in V$)

Prop V an aff. variety

(i) $f, g \in k[x_1, \dots, x_n]$ represent the same poly function on V iff $f - g \in I(V)$

(ii) $(f_1, \dots, f_n)$ and $(g_1, \dots, g_n)$ rep the same poly. mapping $V \to k^n$ iff $f_i - g_i \in I(V)$ $\forall i$.

Proof: (i) If $f - g = h \in I(V)$ $\Rightarrow \forall a \in V$ $h(a) = 0$

$\Rightarrow f(a) - g(a) = 0$

$f(a) = g(a)$ $\forall a \in V$

$\therefore f, g$ rep the same function on V.

If f, g rep same function

$$f(a) - g(a) = 0 \quad \forall a \in V$$

$$\Rightarrow f - g \in I(V) \quad \blacksquare$$

Def) Let $k[V]$ denote the collection of poly. functions $\phi : V \to k$

$\uparrow$ these are equivalence classes

$k[V]$ is a commutative ring

Let $\phi, \psi \in k[V]$

For each $p \in V$ the ring op on $k[V]$ are.

$$(\phi + \psi)(p) = \phi(p) + \psi(p)$$
$$(\phi \cdot \psi)(p) = \phi(p) \cdot \psi(p)$$

Note if f, g irr. $V = V(f \cdot g) = V(f) \cup V(g)$

we can have that $\phi : p \mapsto f(p)$
$\psi : p \mapsto g(p)$

are not identically zero, i.e. $\phi \neq 0 \in k[V]$, $\psi \neq 0$ in $k[V]$.

but $\phi \cdot \psi = 0$

since $f \cdot g \in I(V)$

i.e. $k[V]$ has zero divisors when V is reducible.

Prop) Let $V \subseteq k^n$ be an aff. variety. The following are equivalent:

(i) V is irreducible

(ii) $I(V)$ is a prime ideal

(iii) $k[V]$ is an integral domain

Proof: