Def] Let $f_1, \dots, f_s \in k[x_0, \dots, x_n]$ be homo. poly

we set

$$V(f_1, \dots, f_s) = \left\{ (a_0 : \cdots : a_n) \in \mathbb{P}^n \;\middle|\; \begin{array}{l} f_i(a_0, \dots, a_n) = 0 \\ \forall\, 1 \le i \le s \end{array} \right\}$$

The projective variety defined by $f_1, \dots, f_s$

Dehomgenization

$$U_i = \{ (x_0 : \cdots : x_n) \in \mathbb{P}^n \mid x_i \neq 0 \} \cong k^n$$

If V is a projective variety, is $V \cap U_i$ an affine variety? Yes !! :)

we get defining eqns for $V \cap U_i$ by dehomogenization

Ex] $V \cap U_0$

if $p \in U_0 \Rightarrow p = (1 : x_1 : \cdots : x_n)$

if $f \in k[x_0, \dots, x_n]$ is a defining equation of V

$\Rightarrow$ $g(x_1, \dots, x_n) = f(1, x_1, \dots, x_n)$ vanishes at every point in $V \cap U_i$

These define affine variety

Prop) Let $V = V(f_1, \ldots, f_s)$ homogeneous be a projective variety

Then

$$W = V \cap U_0 \overset{1-1}{\Longleftrightarrow} V(g_1, \ldots, g_s) \subseteq k^n$$

where $g_i(x_1, \ldots, x_n) = f_i(1, x_1, \ldots, x_n)$ for each $1 \le i \le s$

Proof: The discussion above shows we have map

$$\psi : U_0 \longrightarrow k^n$$
$$(1 : x_1 : \cdots : x_n) \longmapsto (x_1, \ldots, x_n)$$

$$\psi(V \cap U_0) \subseteq V(g_1, \ldots, g_s)$$

Since every point in $\psi(V \cap U_0)$ vanish at all g_i's

Take $\hat{p} = (a_1, \ldots, a_n) \in V(g_1, \ldots, g_s)$

$\Rightarrow$ $p = (1 : a_1 : \cdots : a_n) \in U_0$ and

$$f_i(1, a_1, \ldots, a_n) = g_i(a_1, \ldots, a_n) = 0$$

$\therefore$ Since $f_i(p) = 0 \Rightarrow p \in V \cap U_0$

Since the map $\phi : k^n \longrightarrow U_0$

$$(a_1, \ldots, a_n) \mapsto (1 : a_1 : \cdots : a_n)$$

is the inverse of ψ then we have 1-1 map between $V(g_1, \ldots, g_s)$ and $V \cap U_0$. ∎

Ex] Consider $V = (x_1^2 - x_2 x_0 , x_1^3 - x_3 x_0^2) \subseteq \mathbb{P}^3$

To find $V \cap U_0$ we dehomogenize to get

$$V(x_1^2 - x_2 , x_1^3 - x_3) \subseteq k^3$$

Homogeneous Ideal

Suppose $I = (f_1, \ldots, f_s)$ with $f_1, \ldots, f_s$ homogeneous

- I will contain many non-homogeneous polynomials

- Every $g \in I$ vanish on all homog. coords of every $p \in V = V(f_1, \ldots, f_s) \subseteq \mathbb{P}^n$

$$g = \sum A_i f_i$$ and every f_i vanish on all homogeneous coords of p.

Now write A_i as

$$A_i = \sum_{j=1}^{d} A_i^{(j)}$$ ← homogeneous / graded component of degree j

↑ total degree

Sub into expression for g these A_i's

Collecting all homogeneous components we see that all homogeneous components of g live $I = (f_1, \dots, f_s)$

Def) An ideal I in $k[x_0, \dots, x_n]$ is homogeneous if for each $f \in I$ the homogeneous components f_i of f are also in I.

Ex) $I = (\underbrace{y - x^2}_{f}) \leq k[x,y]$

homog. comp. of f are

$f_1 = y$, $f_2 = -x^2$

but neither of f_1 or f_2 are in I.
I is not homog.

Ex $J = (zy - x^2) \leq k[x,y,z]$ is homogeneous

Thm) Let $I \leq k[x_1, \dots, x_n]$ be an ideal. TFAE

(i) I is a homogeneous ideal

(ii) $I = (f_1, \dots, f_s)$ where $f_1, \dots, f_s$ are homogeneous

(iii) A reduced GB of I (w.r.t. to any monomial order) consists of homogeneous poly.

Proof skip

Def For any homogeneous ideal $I \subseteq k[x_0, \ldots, x_n]$ define

$$V(I) = \{ p \in \mathbb{P}^n \mid f(p) = 0 \ \forall f \in I \}$$

Suppose $I = (f_1, \ldots, f_s)$ where $f_1, \ldots, f_s$ is a reduced groebner basis then

$$V(I) = V(f_1, \ldots, f_s)$$

Prop Let $V \subseteq \mathbb{P}^n$ be a projective variety

k an infinite field

$$I(V) = \{ f \in k[x_0, \ldots, x_n] \mid \underbrace{f(a_0, \ldots, a_n) = 0}_{\text{forall representing}} \ \forall \ (a_0 : \cdots : a_n) \in V \}$$

Then $I(V)$ is a homogeneous ideal.