Ideal of a Variety

Let $V \subseteq k^n$

$$I(V) = \{ f \in k[x_1, \ldots, x_n] \mid f(a_1, \ldots, a_n) = 0 \ \forall (a_1, \ldots, a_n) \in V \}$$

Lemma) If $V \subseteq k^n$ is an affine variety then $I(V) \subseteq k[x_1, \ldots, x_n]$ is an ideal

Proof: $\cdot\ 0 \in I(V)$

Let $f, g \in I(V)$, $h \in k[x_1, \ldots, x_n]$, $a = (a_1, \ldots, a_n) \in V$

$$f(a) + g(a) = 0 + 0 = 0$$

$$h(a) f(a) = h(a) \cdot 0 = 0 \quad \therefore \quad I(V) \text{ is an ideal.} \blacksquare$$

$$I(\{(0,0)\}) = (x, y)$$

- Note $f(0,0) = 0 \Rightarrow$ Constant term of f is 0

$f \in (x, y)$

$x - 1 \notin (x, y)$

Ex] $I(k^n) = \{0\}$ (when k is infinite)

$\in k[x_1, \ldots, x_n]$

Q: $I(V(f_1,\dots,f_s)) \stackrel{?}{=} (f_1,\dots,f_s)$?

A: NO, not necessarily.

Lemma: Let $f_1,\dots,f_s \in K[x_1,\dots,x_n]$. Then $(f_1,\dots,f_s) \subseteq I(V(f_1,\dots,f_s))$, equality need not occur.

Proof:

$f \in (f_1,\dots,f_s) \Rightarrow f = \sum_{i=1}^{s} h_i f_i$, $h_i \in K[x_1,\dots,x_n]$

$f_1,\dots,f_s$ all vanish on $V(f_1,\dots,f_s)$ $\therefore$ f does as well

$\therefore \quad f \in I(V(f_1,\dots,f_s))$

$(f_1,\dots,f_s) \subseteq I(V(f_1,\dots,f_s))$

For 2nd part find an example

show $(x^2,y^2) \subsetneq I(V(x^2,y^2))$

$I(V(x^2,y^2))$, $x^2 = y^2 = 0 \Rightarrow V(x^2,y^2) = \{(0,0)\}$

$I(\{(0,0)\}) = (x,y)$

$\therefore I(V(x^2,y^2)) = (x,y)$

and $x \notin (x^2,y^2)$ $\therefore$ $(x^2,y^2) \subsetneq I(V(x^2,y^2))$. ∎

Aside: take square root of each gen. of (x^2, y^2)

we get $I(V(x^2, y^2))$.

Prop Let $V, W \subseteq k^n$ be aff. varieties

(a): $V \subseteq W$ iff $I(V) \supseteq I(W)$

(b): $V = W$ iff $I(V) = I(W)$

Proof: Suppose $V \subseteq W \Rightarrow$ if a poly. vanishes on W it also vanishes on $V \Rightarrow I(W) \subseteq I(V)$

Say $W = V(g_1, \ldots, g_t)$

$\Rightarrow g_1, \ldots, g_t \in I(W) \subseteq I(V)$

$\therefore$ g_i's vanish on V, i.e. any $v \in V$ is in W since W is <u>all</u> common zeros of $g_1, \ldots, g_t$

$\therefore V \subseteq W$.

(b) Follows since $V = W \Leftrightarrow V \supseteq W$ and $V \subseteq W$

$\blacksquare$

Q's :

- Ideal description : can we write all $I \subseteq K[x_1, \dots, x_n]$ as $I = (f_1, \dots, f_s)$? Yes.

- membership : $f \in (f_1, \dots, f_s)$?

- Nullstellensatz : Relationship between $(f_1, \dots, f_s)$ and $I(V(f_1, \dots, f_s))$

Poly. of one var.

Def: $f \in k[x]$

$$f = \underbrace{c_0 x^m}_{\text{Leading term } LT(f) = c_0 x^m} + c_1 x^{m-1} + \dots + c_m \quad , c_i \in k$$

$c_0 \neq 0$, so $\deg(f) = m$.

Ex) $f = \underbrace{7x^5}_{LT(f) = 7x^5} - 4x^4 + 8$

$g = 12x^6 - 27$

$LT(g) = 12x^6$.

- $\deg(f) \leq \deg(g) \iff LT(f) \mid LT(g)$

Division Alg:

Prop: Let