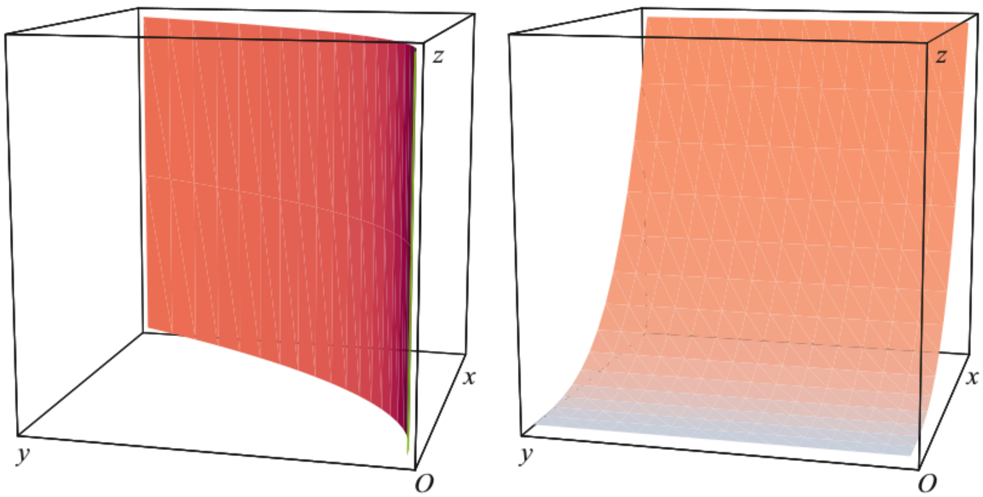
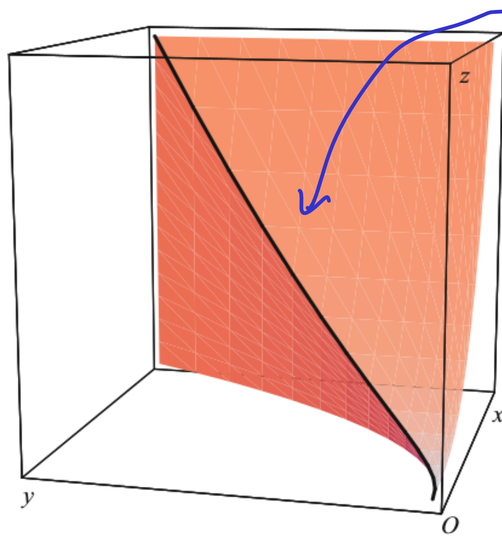
Ex] <u>Twisted cubic</u> $V(y - x^2, z - x^3)$

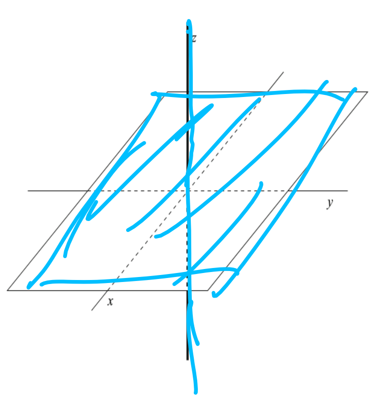




$V(y - x^2, z - x^3)$

Note that the intersection here drops the dim. but this is not always true

Ex] $V(xz, yz) = (x,y)$- plane $\cup$ z-axis



Ex) varieties can also be empty i.e

$V(xy, xy-1)$ is always empty

$xy = xy-1 = 0$ has no solutions

Lemma If $V = V(f_1, \dots, f_s)$, $W = V(g_1, \dots, g_t)$ in K^n

then

$$V \cap W = V(f_1, \dots, f_s, g_1, \dots, g_t) \subseteq K^n$$

$$V \cup W = V(f_i g_j \mid 1 \le i \le s, 1 \le j \le t) \subseteq K^n$$

Proof: $V \cap W \Rightarrow f_1, \dots, f_s$ and $g_1, \dots, g_t$ vanish

$$V \cap W = V(f_1, \dots, f_s, g_1, \dots, g_t)$$

now $V \cup W$

If $a = (a_1, \dots, a_n) \in V \Rightarrow f_i(a) = 0 \therefore f_i(a) g_j(a) = 0$

$$\Rightarrow a \in V(f_i g_j)$$

$$\therefore V \subseteq V(f_i g_j), \quad W \subseteq V(f_i g_j)$$

$$V \cup W \subseteq V(f_i g_j)$$

Take $a \in V(f_i g_j)$ if $a \in V$, we are done

otherwise $f_l(a) \neq 0$ for some l

but $f_l(a) g_j(a) = 0 \quad \forall j$

$$g_j(a) = 0 \quad \forall j$$

$$\therefore \quad a \in W$$

$$V(f_i g_j) \subseteq V \cup W. \qquad \boxtimes$$

Q:

- when is $V(f_1, \ldots, f_s) \neq \emptyset$?
- when is $V(f_1, \ldots, f_s)$ finite ?
- what is $\dim(V(f_1, \ldots, f_s))$?

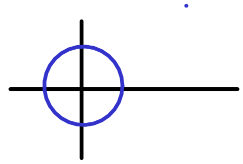
Parameterizations of Aff. Var.

How do we "write-down" solutions to

$$f_1 = \cdots = f_s = 0$$

Finite $\Rightarrow$ list points

Infinite $\Rightarrow$? one way is to parameterize

$$V(x^2+y^2-1) =$$ 

Note point $(-1,0)$ is missing.

or $$x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}$$

rational parametrization

Def: $K(t_1, \dots, t_m)$ is the field of rational functions are elements $\frac{f}{g}$, $f, g \in K[t_1, \dots, t_m]$

$$\frac{f}{g} = \frac{f'}{g'} \quad \text{if} \quad g'f = gf' \in K[t_1, \dots, t_m]$$

For $V = V(f_1, \dots, f_s) \subseteq K^n$

a rational parametric rep. of V consists of $r_1, \dots, r_n \in K(t_1, \dots, t_m)$

s.t

$$x_1 = r_1(t_1, \dots, t_m)$$
$$\vdots$$
$$x_n = r_n(t_1, \dots, t_m)$$

lie in V for all t and want V to be the "smallest" variety containg these points (precise in ch 3)

- may not cover all points
- if $r_1, \dots, r_n$ are poly $\Rightarrow$ poly parametrization

The eqs $f_1 = \dots = f_s = 0$ are called the implict rep. of V.

- Parametrization:
  - Not every variety has a rational param.
    those that do are called unirational
  - It is hard to know if a variety is unirational.

- Implicitization: Given a par. rep. can we always find implicit eq.? Yes !! ☺

Ch 3.

$\boxed{Ex}$

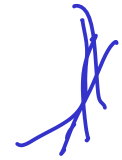
$$x = 1+t$$
$$y = 1+t^2$$

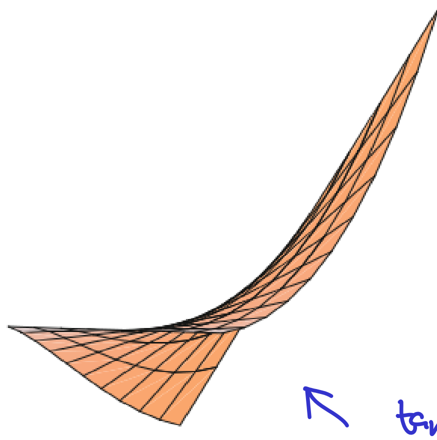
$$t = x-1$$
$$y = 1+(x-1)^2 = x^2-2x+2$$
$$V(y-x^2+2x-2)$$

we have <u>eliminated</u> t,

Ex] twisted cubic $= V(y-x^2, z-x^3)$

Q: what is the surface of tangent lines to all points on the twisted cubic





$\nwarrow$ tangent surface

Note set $x=t$ in $y-x^2 = z-x^3 = 0$

$$x = t$$
$$y = t^2$$
$$z = t^3$$

twist. cubic $= r(t) = (t, t^2, t^3)$

$$r'(t) = (1, 2t, 3t^2)$$

$\Rightarrow$ tangent lines are

$$\underset{\substack{\uparrow \\ \text{Point}}}{r(t)} + \underset{\substack{\uparrow \\ \text{Parameter}}}{u}\ \underset{\substack{\nwarrow \\ \text{Slope}}}{r'(t)} = (t, t^2, t^3) + u(1, 2t, 3t^2)$$

tangent surface $\left\{ \begin{array}{l} x = t+u \\ y = t^2 + 2tu \\ z = t^3 + 3t^2u \end{array} \right. = V\left(x^3 z - \frac{3}{4}x^2y^2 - \frac{3}{2}xyz + y^3 + \frac{z^2}{4}\right)$

Ideals

Def: $I \subseteq K[x_1, \dots, x_n]$ is an ideal if

- $0 \in I$
- $f, g \in I \Rightarrow f+g \in I$
- $hf \in I \quad \forall\, h \in K[x_1, \dots, x_n],\ f \in I$.

Ideal generated by $f_1, \dots, f_s$

$$(f_1, \dots, f_s) = \left\{ \sum_{i=1}^{s} h_i f_i \;\middle|\; \forall\, h_1, \dots, h_s \in K[x_1, \dots, x_n] \right\}$$

- I is <u>finitely generated</u> if $\exists\ f_1, \dots, f_s \in K[x_1, \dots, x_n]$ s.t. $I = (f_1, \dots, f_s)$.

A basis of I.

- Every Ideal in $K[x_1, \dots, x_n]$ is finitely generated

Hilbert basis thm. Ch2.

Prop] If $f_1, \dots, f_s$ and $g_1, \dots, g_t$ are bases of the same ideal in $K[x_1, \dots, x_n]$ so that $(f_1, \dots, f_s) = (g_1, \dots, g_t)$ then we have $V(f_1, \dots, f_s) = V(g_1, \dots, g_t)$

Ex] $(2x^2 + 3y^2 - 11,\ x^2 - y^2 - 3) = (x^2 - 4,\ y^2 - 1) \in K[x_1, \dots, x_n]$.

and

$$V(2x^2 + 3y^2 - 11,\ x^2 - y^2 - 3) = V(x^2 - 4,\ y^2 - 1) = \{(\pm 2, \pm 1)\}$$