Quiz 4 Math 113, Abstract Algebra, Section 3, Spring 2017

Name:

Problem 1. (15 points.) Let $f(x) = x^3 + 11x^2 + 13x + 1$ and let $\mathbb{Q}[x]$ be the polynomial ring in x with rational coefficients. Consider the quotient ring $R = \mathbb{Q}[x]/\langle f(x)\rangle$.

- Is R a field? Is R an integral domain? Prove your answers.

Solution: Yes, R is a field and, hence, also an integral domain.

Proof: We know from a theorem from class (Theorem 16.35 in Judson) that, for S a ring and I an ideal in S, S/I is a field if and only if I is a maximal ideal. We also know from a theorem from class (Theorem 17.22 of Judson) that if F is a field, $p(x) \in F[x]$, then $\langle p(x)\rangle$ is maximal if and only if $p(x)$ is irreducible over F. Hence we may conclude that $R = \mathbb{Q}[x]/\langle f(x)\rangle$ is a field if and only if $f(x)$ is irreducible over $\mathbb{Q}$.

Since $f(x) = x^3 + 11x^2 + 13x + 1$ has degree three it is reducible if and only if it has a linear factor, that is if and only if $f(x)$ has a root in $\mathbb{Q}$. A corollary of Gausss Lemma (Corollary 17.15 of Judson) tells us that if $f(x)$ has a root in $\mathbb{Q}$ it must have a root α in $\mathbb{Z}$, and further, that $\alpha|1$, this would imply that $\alpha = \pm 1$. However $f(1) = 26 \neq 0$ and $f(-1) = -2 \neq 0$, therefore $f(x)$ is irreducible over $\mathbb{Q}$ and $R = \mathbb{Q}[x]/\langle f(x)\rangle$ is a field (and hence an integral domain). ■