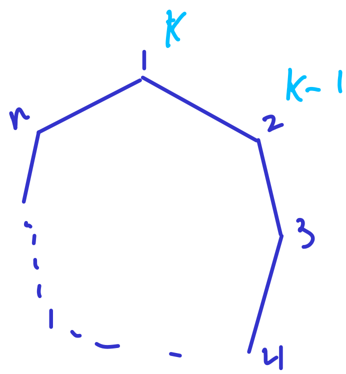
Dihedral groups - subgroups of S_n

n^{th} dihedral group = group of rigid motions of a regular n-gon

= reflections and rotations



This is a subgroup of S_n since we can label the vertices $1, \dots, n$ and consider the default labeling as the identity permutation of $1, \dots, n$

- If we replace 1 by k then 2 must be replaced by $k+1$ or $k-1$
- $2n$ possible rigid motions
  n reflections, n rotations (including the identity)

Thm The dihedral group, D_n, is a subgroup of S_n of order $2n$.

Thm The group D_n $n \geq 3$ consists of all products of two objects r, s satisfying the relations

$r^n = 1$ rotations

$s^2 = 1$ reflections

$srs = r^{-1}$

, where 1 will denote the identity.

Proof: The n- rotations are given geometrically as

$$id, \frac{2\pi}{n}, 2\cdot\frac{2\pi}{n}, \dots, (n-1)\frac{2\pi}{n}$$

$r = \frac{2\pi}{n}$, this generates all other rotations

i.e. $r^k = k\cdot\frac{2\pi}{n}$

Label n reflections $s_1, \dots, s_n$ where s_k leaves the k^{th} vertex fixed.

Even # vertices
- Two vertices fixed by a reflection
- no vertices are fixed

Odd # vertices
- one vertex fixed

In either case $|s_k| = 2$

$s = s_1$, Then $s^2 = 1$, $r^n = 1$

Consider the first vertex of a n-gon

Any motion replacing 1 by k causes 2 to be either $k+1$ or $k-1$.

If $2 \to k+1$ then our transformation was $t = r^{k-1}$

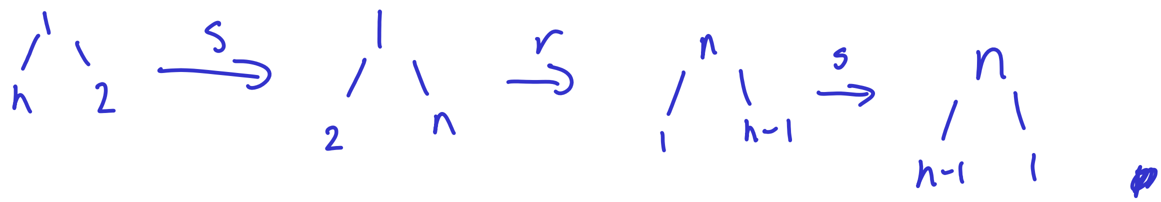
If 2 is replaced by $k-1$

then $t = r^{k-1}s$

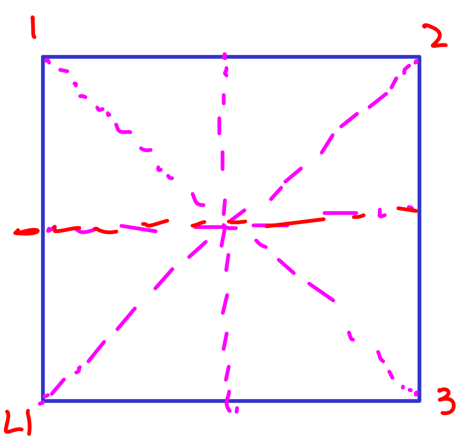
Check $r^{-1} = srs$

i.e. $\frac{-2\pi}{n}$ rotation

$srs =$

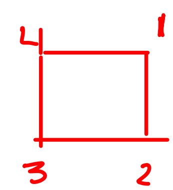


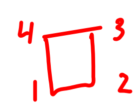
Ex] D_4 rigid motions of a square



$r^3 s_1 = (14)(23)$

$|D_4| = 8$

$r^3 =$ 

$r^3 s_1 =$ 

Rotations

$r = (1\,2\,3\,4)$ = rotation by $\frac{\pi}{2}$ counterclockwise

$r^2 = (13)(24)$ = rotation of π counterclockwise

$r^3 = (1\,4\,3\,2)$ = rotation of $\frac{3\pi}{2}$ counterclockwise

r^4 = identity = (1) = rotation of 0 or 2π.

Reflections

$s_1 = (24)$
$s_2 = (13)$

Two other Reflections

$rs_1 = (12)(34)$
$r^3 s_1 = (14)(23)$

Cosets and Lagrange's Theorem

Cosets

Let G be a group, H a subgroup, $g \in G$

Define a left coset of H with representitive g as

$$gH = \{gh \mid h \in H\}$$

($g+H = \{g+h \mid h \in H\}$ for additive op in G)

right coset

$$Hg = \{hg \mid h \in H\}.$$

Note cosets are not subgroups in general

Ex) Let $H = \langle 3 \rangle = \{0, 3\}$ be the subgroup of $\mathbb{Z}_6$ generated by 3.

$0 + H = 3 + H = \{0, 3\}$

$1 + H = 4 + H = \{1, 4\}$

$2 + H = 5 + H = \{2, 5\}$

Ex] Let K be the subgroup of $S_3 = \{(1), (12), (13), (23), (123), (132)\}$ given by $K = \{(1), (12)\}$

left cosets

$(1)K = (12)K = \{(1), (12)\} = K(1) = K(12)$

$(13)K = (123)K = \{(13), (123)\}$

$(23)K = (132)K = \{(23), (132)\}$

$[G : K] = 3$

Right cosets

$K(13) = K(132) = \{(13), (132)\}$

$K(23) = K(123) = \{(23), (123)\}$

and $K(1)$

3 unique cosets (right or left)

Lemma: Let H be a subgroup of G a group. $g_1, g_2 \in G$
The Following are equivalent:

(1) $g_1 H = g_2 H$ $\rightarrow$ $\forall\, h_1 \in H$ $\exists\, h_2 \in H$ s.t. $g_1 h_1 = g_2 h_2$

$\Rightarrow h_1 = g_1^{-1} g_2 h_2$

$\Rightarrow h_1 h_2^{-1} = g_1^{-1} g_2$

$\Rightarrow g_1^{-1} g_2 \in H$

(2) $H g_1^{-1} = H g_2^{-1}$

(3) $g_1 H \subseteq g_2 H$

(4) $g_2 \in g_1 H$

(5) $g_1^{-1} g_2 \in H$

Thm) Let H be a subgroup of a group G. The left (resp. right) cosets of H in G partition G. That is G is the disjoint union of the left (or right) cosets of H in G.

Proof:

Let $g_1 H$, $g_2 H$ be two cosets of H in G. Show that:

$$g_1 H \cap g_2 H = \emptyset \quad \underline{\text{or}} \quad g_1 H = g_2 H.$$

Suppose $g_1 H \cap g_2 H \neq \emptyset$ and $a \in g_1 H \cap g_2 H$ then

$$a = g_1 h_1 = g_2 h_2 \quad \text{for some } h_1, h_2 \in H$$

$$g_1 = g_2 \underbrace{\overbrace{h_2 h_1^{-1}}^{\in H}}_{= \tilde{h}} \Rightarrow g_1 \in g_2 H$$

$$g_1 H = \{ g_1 h \mid h \in H \} = \{ g_2 \tilde{h} \cdot h \mid h \in H \}$$

$$= \{ g_2 \hat{h} \mid \hat{h} \in H \}$$

$$= g_2 H.$$

And all $g \in G$ appear in some coset, in particul in gH since $g \cdot e = g$ and $e \in H$. ∎