Things to know for midterm

- Cyclic subgroups
- Groups, Subgroups
- Normal subgroups
- cosets
- Factor groups
- isomorphisms
- homomorphisms
- internal/external direct product

Ex/ $T = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \;\middle|\; ac \neq 0, a,b,c \in \mathbb{R} \right\}$

$U = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}$

Show T/U abelian:

$g \in T$ an element of T/U has for gU

Claim $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} U = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} U$

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & \frac{-b}{a} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & a\left(\frac{-b}{a}\right) + b \\ 0 & c \end{pmatrix}$$

$\in \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} U$

$$= \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\in U}$$

$\in \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} U$

since $\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \in$

$\therefore \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} U \cap \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} U \neq \phi \Rightarrow \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} U = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} U.$

$$gH \cap kH \neq \emptyset \iff gH = kH$$

$$G = \coprod gH$$

all unique cosets

$$\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix} U \cdot \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} U = \begin{pmatrix} a_1 & 0 \\ 0 & c_1 \end{pmatrix} U \cdot \begin{pmatrix} a_2 & 0 \\ 0 & c_2 \end{pmatrix} U$$

$$= \begin{pmatrix} a_2 & 0 \\ 0 & c_2 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & c_1 \end{pmatrix} U$$

$$= \begin{pmatrix} a_2 & 0 \\ 0 & c_2 \end{pmatrix} U \begin{pmatrix} a_1 & 0 \\ 0 & c_1 \end{pmatrix} U .$$

$$= \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} U \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix} U .$$

Hom Ex/

Consider $\phi : \mathbb{R} \longrightarrow \mathbb{C}^*$

$: \theta \longmapsto \cos\theta + i\sin\theta$

Check this is a hom.

$$\phi(\alpha+\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

$$= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$$

$$= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

$$= \phi(\alpha)\phi(\beta)$$

$$\ker(\phi) = \phi^{-1}(1) = \{2\pi n \mid n \in \mathbb{Z}\}$$

Everything that sends $\cos \to 1$, $\sin \to 0$

$\ker(\phi) \simeq \mathbb{Z}$ by $\psi(2\pi n) = n$

Ex | Determine all hom from

$$\mathbb{Z}_7 \to \mathbb{Z}_{12}$$

- $\ker \phi$ is always a subgroup of $\mathbb{Z}_7$
- $\phi(H)$, H a subgroup of $\mathbb{Z}_7$, is a subgroup of $\mathbb{Z}_{12}$

The only subgroups of $\mathbb{Z}_7$ are $\{0\}$, $\mathbb{Z}_7$

$$\ker \phi = \{0\} \text{ or } \mathbb{Z}_7$$

but $\mathbb{Z}_{12}$ has no subgroups of order 7 is a hom.

$\therefore$ ϕ cannot be 1-1

$\therefore$ $\ker(\phi) \neq \{0\}$

$\therefore$ $\ker(\phi) = \mathbb{Z}_7$

$$\phi : n \mapsto 0$$

is the only homomorphism.

Note: If ϕ is not 1-1

$\phi(a) = \phi(b) = g$ for some $a \neq b$

$\Rightarrow$ $\phi(a)\phi(b^{-1}) = g\,g^{-1}$

$\Rightarrow$ $\phi(ab^{-1}) = e$

$\therefore$ $ab^{-1} \in \ker\phi$

$\therefore$ if ϕ is not 1-1

$\Rightarrow$ $|\ker \phi| > 1$

Ex | $\phi : G \to H$ is a hom. G abelian show $\phi(G)$ is abelian.

Proof: i.e. show, if $a = \phi(g_1)$, $b = \phi(g_2)$, that $ab = ba$

$$ab = \phi(g_1)\phi(g_2) = \phi(g_1 g_2) = \phi(g_2 g_1) = \phi(g_2)\phi(g_1)$$
$$= ba \quad \#$$

Ex] If $G = \langle a \rangle$

$$\phi(a^n) = \phi(a \cdots a) = \phi(a) \cdots \phi(a) = \phi(a)^n$$

$$\therefore \phi(G) = \langle \phi(a) \rangle$$

Ex] Prove or disprove. If H and G/H are cyclic $\Rightarrow$ G is cyclic.

$G = \mathbb{Z}_2 \times \mathbb{Z}_2$

$H = \langle (1,0) \rangle$,

$G/H \cong \langle (0,1) \rangle$

$\| $

$\{ (0,0)H, (0,1)H \}$ But $\mathbb{Z}_2 \times \mathbb{Z}_2$ is <u>not</u> cyclic.

If G/H, H are abelian $\Rightarrow$ G abelian

Counter Example $T =$ upper triang invertible (see above)

T/U is abelian

OR

Quaternions

$Q = \{ \pm 1, \pm i, \pm j, \pm k \}$

$H = \{ \pm 1, \pm i \} \cong \mathbb{Z}_4$

$Q/H \cong \mathbb{Z}_2$

Since $|Q/H| = 2$

Since $|Q| = 8$

$|H| = 4$

$\therefore [Q : H] = 2.$