Factor Groups = group of cosets of a normal subgroup

Def: Let N be a normal subgroup of G. The factor group or quotient group G/N is a group consisting of cosets of N in G where the operation is given by $(aN)(bN) = abN$, where $aN, bN \in G/N$ $(a, b \in G)$

Thm: Let G be a group, N a normal subgroup of G. The cosets of N in G form a group G/N of order $[G:N]$ (with op $(aN)(bN) = abN$)

Proof: First we must show the group op. is well defined (i.e. that different rep. of the same coset yeild the same result)

Suppose $aN = bN \in G/N$

and $cN = dN \in G/N$

we must show $(aN)(cN) = (bN)(dN)$

By def $(aN)(cN) = acN$

Since $aN = bN$ and $cN = dN$ $\Rightarrow \exists\, n_1, n_2 \in N$ s.t $a = bn_1$, $c = dn_2$

$(aN)(cN) = acN = bn_1 d \underbrace{n_2 N}$ $n_2 N = N$ since $n_2 \in N$

$= b n_1 d N$

$\updownarrow$ N is a normal subgroup

$= b \widetilde{n_1 N} d$ $n_1 \in N$

$= b N d$

$\updownarrow$ N is normal

$= b d N = (bN)(dN)$

$\therefore$ this group op. is well defined

$\therefore$ G/N is a set with a well defined associative binary operation

$(aN\, bN)\, cN = aN\,(bN\, cN)$

identity : $eN = N$

inverses : $(gN)^{-1} = g^{-1} N$

$[G:N] =$ # of cosets of N in G

and G/N is all cosets of N in G

$\therefore$ $|G/N| = [G:N]$ $\blacksquare$

Ex] we know one factor group :)

$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$

- Any subgroup $n\mathbb{Z}$ is normal in $\mathbb{Z}$ (since $\mathbb{Z}$ is Abelian)

$$\left.\begin{array}{l} 0 + n\mathbb{Z} = n\mathbb{Z} \\ 1 + n\mathbb{Z} \\ \vdots \\ (n-1) + n\mathbb{Z} \end{array}\right\}$$

These are the elements of $\mathbb{Z}/n\mathbb{Z}$ and also the elements of $\mathbb{Z}_n$

$\parallel$

$\{0, 1, \ldots, n-1\}$

$[0] = \{\ldots, -n, 0, n, \ldots\}$

$\mathbb{Z}_5 = \mathbb{Z}/5\mathbb{Z}$

$0 + 5\mathbb{Z} = \{ \ldots, -5, 0, 5, 10, \ldots \}$

$\quad\quad\quad\quad \shortparallel$ eq class of $0 \bmod 5$

$1 + 5\mathbb{Z} = \{ \ldots, -4, 1, 6, 11, \ldots \}$

$\quad\quad\quad\quad \shortparallel$ eq. class of $1 \bmod 5$

$j \equiv k \bmod n$ iff $j - k = vn$ for some $v \in \mathbb{Z}$

this is precisely the def of $j - k \in n\mathbb{Z}$

Ex) $N = \{(1), (132), (123)\}$ (set of even permutations) is a normal subgroup of S_3. Write mult table for S_3/N

$S_3/N = \{N, (12)N\}$ ($(12)N$: set of odd perm)

^ Since $[S_3 : N] = 2$

	N	$(12)N$
N	N	$(12)N$
$(12)N$	$(12)N$	N

$\therefore$ S_3/N is a cyclic group of order 2 (generated by $(12)N$) $\therefore$ $S_3/N \cong \mathbb{Z}_2$

Ex) D_n - dihedral group gen by r, s, s.t.

$$r^n = id$$
$$s^2 = id$$
$$srs = r^{-1}$$
$$\hookrightarrow \quad sr = r^{-1}s$$

$\langle r \rangle$ = cyclic subgroup of rotations

$$|\langle r \rangle| = n$$

$N = \langle r \rangle$ is a normal subgroup

So show $gNg^{-1} \subseteq N$, show for arbitrary $g \in D_n, n \in N$ $\; gng^{-1} \in N$

if $g \in D_n \Rightarrow g \in N$ or $g = sr^k$ or $r^k s$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\; \overset{\shortparallel}{sr^{-k}}$

$\Rightarrow$ if $g \in N$ $\quad gng^{-1} \in N$

since $srs = r^{-1}$

or if $g = sr^{l}$, $n = r^k \in N$

$$sr^l r^k (sr^l)^{-1} = sr^{l+k} r^{-l} s^{-1}$$
$$= s r^k s = r^{-k} \in N$$

$$|D_n / \langle r \rangle| = [D_n : \langle r \rangle] = 2$$

$$\therefore \quad D_n / \langle r \rangle = \{ \langle r \rangle, s\langle r \rangle \} \simeq \mathbb{Z}_2$$

Group Homomorphisms

Let $(G, \cdot)$, $(H, \circ)$ be groups

$\phi : G \longrightarrow H$ is a <u>homomorphism</u>

if it preserves the group operation i.e.

$$\phi(g_1 g_2) = \phi(g_1) \circ \phi(g_2) \quad \forall\, g_1, g_2 \in G.$$

Ex] G - group, $g \in G$

$$\phi : \mathbb{Z} \to G$$
$$n \mapsto g^n \quad \text{is a group homomorphism}$$

Since $\phi(m+n) = g^{m+n} = g^m g^n = \phi(m)\phi(n)$

ϕ maps onto $\langle g \rangle$, but Prob. not onto G

Ex] $G = GL_2(\mathbb{R})$ = 2×2 inventible matrices with mult.

$$\phi : GL_2(\mathbb{R}) \to \mathbb{R}^*$$
$$A \mapsto \det(A)$$

Since

$$\phi(AB) = \det(AB) = \det(A)\det(B) = \phi(A)\phi(B)$$

(and $\det(A) \neq 0 \quad \forall\, A \in GL_2(\mathbb{R}) \therefore \phi(A) \in \mathbb{R}^* \quad \forall\, A \in GL_2(\mathbb{R})$)

Prop] Let $\phi : G_1 \to G_2$ be a hom. of groups.

Then:

1) If $e \in G_1$ is the identity the $\phi(e)$ is the identity in G_2
2) $\forall\, g \in G_1 \quad \phi(g^{-1}) = (\phi(g))^{-1}$
3) H_1 subgroup of $G_1 \Rightarrow \phi(H_1)$ is a subgroup of G_2
4) H_2 is a subgroup of $G_2 \Rightarrow \phi^{-1}(H_2) = \{ g \in G_1 \mid \phi(g) \in H_2 \}$

is a subgroup of G_1. If H_2 is a normal subgroup in G_2 then $\phi^{-1}(H_2)$ is normal in G_1.

Proof:

e = id in G_1, $\tilde{e}$ = id in G_2

1) $\tilde{e}\,\phi(e) = \phi(e) = \phi(ee) = \phi(e)\phi(e)$

$\Rightarrow \tilde{e} = \phi(e)$

2) $\phi(g^{-1})\phi(g) = \phi(g^{-1}g) = \phi(e) = \tilde{e}$

$\tilde{e} = \phi(g)\phi(g^{-1})$

$\therefore \phi(g^{-1}) = (\phi(g))^{-1}$

3) Show $\phi(H_1)$ is a subgroup of G_2

$e \in H_1 \Rightarrow \phi(e) = \tilde{e} \in \phi(H_1)$ $\therefore \phi(H_1)$ is non-empty and contains $\tilde{e} \in G_2$

Let $x, y \in \phi(H_1) \Rightarrow \exists\, a, b \in H_1$ s.t. $\phi(a) = x$, $\phi(b) = y$

Show $xy^{-1} \in \phi(H_1)$

$$xy^{-1} = \phi(a)(\phi(b))^{-1} = \phi(ab^{-1}) \in \phi(H_1)$$

$\therefore \phi(H_1)$ is a subgroup of G_2.

4) H_2 subgroup of G_2, Consider the set $H_1 = \phi^{-1}(H_2)$

$e \in \phi^{-1}(H_2)$ since $\tilde{e} = \phi(e) \Rightarrow \phi^{-1}(\tilde{e}) = \{e\}$

If $a, b \in H_1 \Rightarrow \phi(ab^{-1}) = \phi(a)\,\phi(b^{-1}) = \underbrace{\overset{\in H_2}{\phi(a)}\,(\overset{\in H_2}{\phi(b)})^{-1}}_{\in H_2}$

$\Downarrow$ $\qquad\qquad$ $\uparrow$ by this

For any $a, b \in H_1 \Rightarrow ab^{-1} \in H_1 = \phi^{-1}(H_2)$

$\therefore$ H_1 is a subgroup of G_1

If H_2 is normal in G_2

Show $g^{-1}hg \in H_1 \quad \forall\, h \in H_1 = \phi^{-1}(H_2) \quad \forall\, g \in G_1$

$$\phi(g^{-1}hg) = \underbrace{\phi(g)^{-1}\,\overset{\in H_2}{\phi(h)}\,\phi(g)}_{\in H_2 \text{ since } H_2 \text{ is normal}}$$

$$\Rightarrow g^{-1}hg \in \phi^{-1}(H_2) = H_1$$

$\therefore$ $\phi^{-1}(H_2)$ is normal

Defl Let $\phi: G \to H$ a group hom. $e \in H$ the id. in H

The kernel of ϕ is the subgroup of G

given by

$$\ker(\phi) = \phi^{-1}(\{e\}) = \{g \in G \mid \phi(g) = e\}$$

Thm) $\ker(\phi)$ is a normal subgroup of G.