Ex] D_6 is the internal direct product

$H = \{1, r^3\}$, $K = \{1, r^2, r^4, s, r^2s, r^4s\}$ (s: reflection in 3rd vertex)

- $H \cap K = \{1\}$
- $HK = D_6$
- $hk = kh \quad \forall\, h \in H, k \in K$

$$D_6 \simeq \mathbb{Z}_2 \times S_3$$

Ex] S_3 can not be written as an internal direct product

$|S_3| = 6 \Rightarrow |H| = 3$, $|K| = 2$

The only subgroup of S_3 of order 3

$H = \{(1), (123), (132)\}$

All subgroups of order 2 are $=\{id, \text{2-cycle}\}$

but (2-cycle)(3-cycle) $\neq$ (3-cycle)(2-cycle) in S_3

$(12)(123) = (23)$

$(123)(12) = (13)$

Also check

$(13)(123) \neq (123)(13)$

and $(23)(123) \neq (123)(23)$

$\therefore$ any subgroup of order 2 always has an element which does not commute with $(123) \in H$.

$\therefore$ S_3 is not an internal dircet product.

Thm Let G be the internal direct product of subgroups H and K. Then $G \simeq H \times K$.

Proof: G is an internal direct product $\Rightarrow$ $\forall g \in G$

$$g = hk \quad , h \in H \, , \, k \in K$$

this h, k

Define $\phi : G \longrightarrow H \times K$

$$g \longmapsto (h, k)$$

Show ϕ is well defined, need h, k to uniquely determine g.

$$g = hk = h'k'$$

Consider

$$hk = h'k'$$

$$\cancel{h^{-1}} \underbrace{hk(k')^{-1}}_{\in K} = \underbrace{h^{-1}h'}_{\in H} \cancel{k'(k')^{-1}}$$

$k(k')^{-1} = h^{-1}h' = e$ since $H \cap K = \{e\}$

$\Downarrow$

$k(k')^{-1} = e \Rightarrow k = k'$

and $h^{-1}h' = e \Rightarrow h = h'$

Show ϕ preserves group op.

$$g_1 = h_1 k_1 \, , \quad g_2 = h_2 k_2$$

$$\phi(g_1 g_2) = \phi(h_1 k_1 \cdot h_2 k_2)$$

h's and k's commute

$$= \phi(h_1 h_2 k_1 k_2)$$

$$= (h_1 h_2 , k_1 k_2)$$

$$= (h_1, k_1) \cdot (h_2, k_2) = \phi(g_1)\phi(g_2)$$

1-1, onto homework.

Ex) $\mathbb{Z}_6 \simeq \{0,2,4\} \times \{0,3\} \simeq \mathbb{Z}_3 \times \mathbb{Z}_2$

$\mathbb{Z}_3 = \{0,1,2\}$, $\mathbb{Z}_2 = \{0,1\}$

For a collection of subgroups $H_1, \dots, H_n$ of G, G is the internal direct product of $H_1, \dots, H_n$

$$G = H_1 H_2 \cdots H_n = \{h_1 h_2 \cdots h_n \mid h_i \in H_i\}$$

$$H_i \cap \left(\bigcup_{i \neq j} H_j \right) = \{e\}$$

$$h_j h_i = h_i h_j \quad \forall h_i \in H_i \ \forall h_j \in H_j$$

Thm $G \simeq H_1 \times H_2 \times \cdots \times H_n$.

Factor Groups and Normal subgroups

subgroups N of G s/t $gN = Ng \ \forall g \in G$

Build a group out of cosets

$\mathbb{Z}_5 = \mathbb{Z}/5\mathbb{Z}$

Def A subgroup N of a group G is a normal subgroup of G if $gN = Ng \quad \forall g \in G$

normal subgroup = same right and left cosets for all elements of g.

Ex] Every subgroup of an abelian group is normal since all elements commute.

Ex] $H = \{(1), (12)\}$ in S_3

$$(123)H = \{(123), (13)\} \neq H(123) = \{(123), (23)\}$$

$\therefore$ H is not a normal subgroup

$N = \{(1), (123), (132)\}$ is normal

$$(12)N = N(12) = \{(12), (13), (23)\}$$

$\therefore$ N is a normal subgroup of S_3.

Normal subgroup test

Thm] G = group. N a normal subgroup of G.

The following are equivilent:

1) N is a normal subgroup in G
2) $\forall g \in G \quad gNg^{-1} \subseteq N$
3) $\forall g \in G \quad gNg^{-1} = N$

Proof: $(1) \Rightarrow (2)$ N is normal in $G \Rightarrow gN = Ng \quad \forall g \in G$

For a fixed (arbitrary) $g \in G$ $n \in N$ $\exists\, \hat{n} \in N$ s.t.

$$gn = \hat{n} g$$

$$gng^{-1} = \hat{n} \in N \Rightarrow gNg^{-1} \subseteq N$$

2) $\Rightarrow$ 3) Let $g \in G$ be arbitrary (fixed)

Suppose $gNg^{-1} \subseteq N$ $\forall g \in G$, Show $N \subseteq gNg^{-1}$ (i.e. $N = gNg^{-1}$)

$gNg^{-1} \subseteq N$

$$\Rightarrow g^{-1} N (g^{-1})^{-1} \subseteq N$$

$$\Rightarrow g^{-1} n g \in N \quad \forall g \in G, \forall n \in N$$

$$\Rightarrow g^{-1} n g = \hat{n} \quad \text{for some } \hat{n} \in N$$

and $\forall g \in G, \forall n \in N$

$$n = g \hat{n} g^{-1}$$

for all $g \in G$

$$\therefore \quad n \in gNg^{-1}$$

$$N \subseteq gNg^{-1}$$

$$\Rightarrow N = gNg^{-1}.$$

(3) $\Rightarrow$ (1) Suppose $gNg^{-1} = N$ $\forall g \in G$

$$\Rightarrow \forall n \in N \ \exists \ \hat{n} \in N \text{ s.t. } gng^{-1} = \hat{n}$$

$$\Rightarrow \quad gn = \hat{n} g \quad \forall g \in G, \forall n \in N$$

$$\therefore \quad gN \subseteq Ng$$

Similarrily $\forall g \in G$, $\forall \hat{n} \in N$ $\exists$ $n \in N$

$$Ng \subseteq gN \Rightarrow gN = Ng.$$

Factor Groups = group of cosets of a normal subgroup

Defn: Let N be a normal subgroup of G. The factor group or quotient group G/N is a group consisting of cosets of N in G where the operation is given by $(aN)(bN) = abN$, where $aN, bN \in G/N$ $(a, b \in G)$

Thm: Let G be a group, N a normal subgroup of G. The cosets of N in G form a group G/N of order $[G:N]$ (with op)

Proof: First we must show the group op. is well defined (i.e. that different rep. of the same coset yeild the same result)

Suppose $aN = bN \in G/N$

and $cN = dN \in G/N$

we must show $(aN)(cN) = (bN)(dN)$