16. If $|G| = 2n$, prove that the number of elements of order 2 is odd. Use this result to show that G must contain a subgroup of order 2.

Proof:

Let $A_1 = \{a \in G \mid |a| = 2\}$

[this is a subset (not a subgroup) but, if A_1 is non-empty then each $a \in A_1$ generate a subgroup $\langle a \rangle$ of order 2.]

And let $A_2 = \{a \in G \mid |a| > 2\}$.

write G as $G = \{e\} \cup A_1 \cup A_2$.

Note if $|a| = 2$, $a \in G$

$\Rightarrow a^2 = e \Leftrightarrow a = a^{-1}$

So $a = a^{-1}$ if and only if $|a| = 2$ or $a = e$.

This means that for $b \in A_2$ we have $b \neq b^{-1}$.

Note that if $b_1 \neq b_2 \Rightarrow b_1^{-1} \neq b_2^{-1}$ (since inverses are unique)

For any $b \in G$ $|b| = |b^{-1}|$ [Since $b^j = e \Rightarrow (b^{-1})^j = (b^j)^{-1} = e^{-1} = e$]

$\therefore$ A_2 contains an even number of elements since it must consist of pairs b, b^{-1} where $b \neq b^{-1}$.

$\therefore$ $\{e\} \cup A_2$ has an odd number of elements, but

$G = \{e\} \cup A_1 \cup A_2$ has an even number of elements $\Rightarrow A_1$ has an odd number of elements $\therefore$ there are an odd number of $a \in G$ s.t. $|a| = 2$

$\Rightarrow$

show that G must contain a subgroup of order 2.

17. Suppose that $[G : H] = 2$. If a and b are not in H, show that $ab \in H$.

Proof:

$[G:H]=2 \Rightarrow$ By definition, there are two cosets of H in G. Suppose $a \notin H$, $b \notin H$, then $aH \neq H$, $bH \neq H$

also note that if $a \notin H$ then $a^{-1} \notin H$, since if $a^{-1} \in H \Rightarrow (a^{-1})^{-1} = a \in H$ since H is a group. Since there are two cosets of H in G, and $eH = H$ must be one of these we have

$$G = H \sqcup aH = H \sqcup bH = H \sqcup a^{-1}H$$

Thus $a^{-1}H = bH$

$\Rightarrow \quad H = (a^{-1})^{-1}bH \quad \Rightarrow \quad H = abH$

$\therefore h = ab \cdot e$ for some $h \in H$

$\therefore ab \in H$. ∎