## §11.3

16. If $H$ and $K$ are normal subgroups of $G$ and $H \cap K = \{e\}$, prove that $G$ is isomorphic to a subgroup of $G/H \times G/K$

*Proof.* Define a homomorphism $\varphi : G \to G/H \times G/K$ specified by $\varphi : g \mapsto (gH, gK)$. To see this is a homomorphism note that for $g_1, g_2 \in G$

$$\varphi(g_1)\varphi(g_2) = (g_1H, g_1K)(g_2H, g_2K) = (g_1g_2H, g_1g_2K) = \varphi(g_1g_2).$$

By the First Isomorphism Theorem we have that

$$G/\ker(\varphi) \cong \varphi(G), \tag{1}$$

further from a theorem in class (Proposition 11.4 in our book) we know that $\varphi(G)$ is a subgroup of $G/H \times G/K$. To conclude we must calculate $\ker(\varphi)$; recall the identity in $G/H \times G/K$ is $(H, K)$. By definition

$$\ker(\varphi) = \{g \in G \mid \varphi(g) = (H, K)\} = \{g \in G \mid (gH, gK) = (H, K)\}.$$

From this we see that for all $g \in \ker(\varphi)$ we have that $gH = H$ and $gK = K$, hence we must have that $g \in H$ and $g \in K$. Using this we may express the kernel as

$$\ker(\varphi) = \{g \in G \mid g \in H \text{ and } g \in K\} = H \cap K.$$

Further, from the statement of the question, we have that $H \cap K = \{e\}$. With this we may rewrite (1) as

$$G/(H \cap K) = G/\{e\} = G \cong \varphi(G),$$

hence $G$ is isomorphic to a subgroup of $G/H \times G/K$. □