§9.3

26. Let $\varphi : G \to H$ be a group isomorphism. Show that $\varphi(x) = e_H$ if and only if $x = e_G$, where e_G and e_H are the identities of G and H, respectively.

    First suppose $\varphi(x) = e_H$, then $\varphi(x)(\varphi(x))^{-1} = e_H(e_H)^{-1} = e_H$, since φ is a isomorphism then $\varphi(x)(\varphi(x))^{-1} = \varphi(xx^{-1}) = \varphi(e_G) = e_H$. Therefore $e_H = \varphi(x) = \varphi(e_G)$, which implies that $x = e_G$.

    Now suppose that $x = e_G$ and let $h = \varphi(g) \in H$ (such a g exists for all $h \in H$ since φ is an isomorphism). Then

    $$h\varphi(x) = \varphi(g)\varphi(x) = \varphi(gx) = \varphi(ge_G) = \varphi(g) = h = \varphi(e_G g) = \varphi(x)\varphi(g) = \varphi(x)h,$$

    and hence $\varphi(x) = e_H$ be definition. □

47. If $G \cong \bar{G}$ and $H \cong \bar{H}$ show that $G \times H \cong \bar{G} \times \bar{H}$.

    Let θ denote the isomorphism $G \to \bar{G}$ and let ψ denote the isomorphism $H \cong \bar{H}$. Define a map $\phi : G \times H \to \bar{G} \times \bar{H}$ specified by

    $$\phi : (g, h) \mapsto (\theta(g), \psi(h)) \quad \forall g \in G, h \in H.$$

    First show ϕ is 1-1. Suppose that $\phi(g_1, h_1) = \phi(g_2, h_2)$, then by the definition of ϕ we have that $(\theta(g_1), \psi(h_1)) = (\theta(g_2), \psi(h_2))$, this means that $\theta(g_1) = \theta(g_2)$ and $\psi(h_1) = \psi(h_2)$. Since θ and ψ are bijective this means that $(g_1, h_1) = (g_2, h_2)$ and hence ϕ is 1-1. Again since θ and ψ are bijective, for any $\bar{g} \in \bar{G}$ and any $\bar{h} \in \bar{H}$ we have that $\bar{g} = \theta(g)$ and $\bar{h} = \psi(h)$, hence for all $(\bar{g}, \bar{h}) \in \bar{G} \times \bar{H}$ we have that $(\bar{g}, \bar{h}) = (\theta(g), \psi(h))$ for some $g \in G$, $h \in H$. To see that ϕ is a homomorphism note that

    $$\begin{aligned}\phi(g_1, h_1)\phi(g_2, h_2) =& (\theta(g_1), \psi(h_1))(\theta(g_2), \psi(h_2)) \\ =& (\theta(g_1)\theta(g_2), \psi(h_1)\psi(h_2)) \\ =& (\theta(g_1 g_2), \psi(h_1 h_2)) \\ =& \phi((g_1 g_2, h_1 h_2)) \\ =& \phi((g_1, h_1)(g_2, h_2)).\end{aligned}$$

    □