Show that $V \cong V^{**}$:

Proof:

We know (from (a), (b)) that V^* is the space of linear maps (also called functionals)

For a v.space W over a feild F a linear functional is a map $\phi: W \longrightarrow F$ which is linear (as described in (a))

Hence we want to first see what the elements of $(V^*)^*$ are

(and it would be good to relate them to V)

For each $v \in V$ we may define a linear functional on V^* (that is an element of $(V^*)^*$) by evaluation, i.e.

$$\lambda_v : V^* \to F$$
$$\phi \mapsto \phi(v)$$

$\phi \in V^*$, that is $\phi: V \longrightarrow F$.

First show λ_v is a linear functional, that is verify λ_v is a v.space homomorphism from V^* to F: , let $v \in V$ be arbitrary, $\phi_1, \phi_2 \in V^*$, $a \in F$

$$\lambda_v(\phi_1 + \phi_2) = (\phi_1 + \phi_2)(v) \underset{\text{By def. from (a)}}{=} \phi_1(v) + \phi_2(v) = \lambda_v(\phi_1) + \lambda_v(\phi_2).$$

$$\lambda_v(a\phi_1) = a\,\phi_1(v) = a\cdot(\lambda_v(\phi_1))$$

$\therefore$ λ_v is a v. space hom. for any $v \in V$.

Note λ_v also allows us to define a map

$$\psi : V \longrightarrow (V^*)^*$$
$$v \mapsto \lambda_v$$

this is a functional on V^* by above $\therefore$ it is in $(V^*)^*$.

Now all that remains is to show that ψ is an isomorphism.

First show ψ is a homomorphism of V. spaces. Let $v_1, v_2 \in V$ and fix an arbitrary $\phi \in V^*$

Show $\psi(v_1+v_2)(\phi) = \psi(v_1)(\phi) + \psi(v_2)(\phi) \quad \forall \phi \in V^*$

[this shows $\psi(v_1+v_2) = \psi(v_1) + \psi(v_2)$]

$$\psi(v_1+v_2)(\phi) = \lambda_{v_1+v_2}(\phi) = \phi(v_1+v_2) = \phi(v_1)+\phi(v_2)$$
$$= \lambda_{v_1}(\phi) + \lambda_{v_2}(\phi)$$
$$= \psi(v_1)(\phi) + \psi(v_2)(\phi)$$

Similarily

$$\psi(cv_1)(\phi) = \lambda_{cv_1}(\phi) = \phi(cv_1) = c\,\phi(v_1) = c\lambda_{v_1}(\phi)$$
$$= c\,\psi(v_1)(\phi)$$

$\therefore$ ψ is a v. space hom.

By (b) any hom $V \to (V^*)$ must be surjective since $\dim(V) = \dim(V^*)$ since both have the same number of vectors in their basis,

$\therefore$ $\dim(V) = \dim(V^{**})$ since $(V^*)^* = V^{**}$

it remains to show ψ is 1-1, i.e. $\ker\psi = 0$. Let $v \in V$

If $\psi(v) = 0 \Rightarrow \lambda_v = 0 \Rightarrow \phi(v) = 0 \quad \forall \phi \in V^*$

if $v \neq 0$ then there must be some coefficient α_i of some basis vector of V, b_i, such that $\alpha_i \neq 0$.

But by using the basis of V^* given in (b) then

$\phi_i(\alpha_i b_i) = \alpha_i \neq 0 \quad \therefore$ if $\phi(v) = 0 \;\forall \phi \in V^*$ then $v = 0$

ϕ_i: basis of V^* from (b)

$\therefore \ker\psi = 0 \quad \therefore$ ψ is an isomorphism and $V \cong V^{**}$. ■