Rings

A non-empty set R is a ring if it is closed under 2 binary operations: addition and multiplication with the following. $a,b,c \in R$

Abelian group $(R,+)$ with addition and an associative + distributive mult.

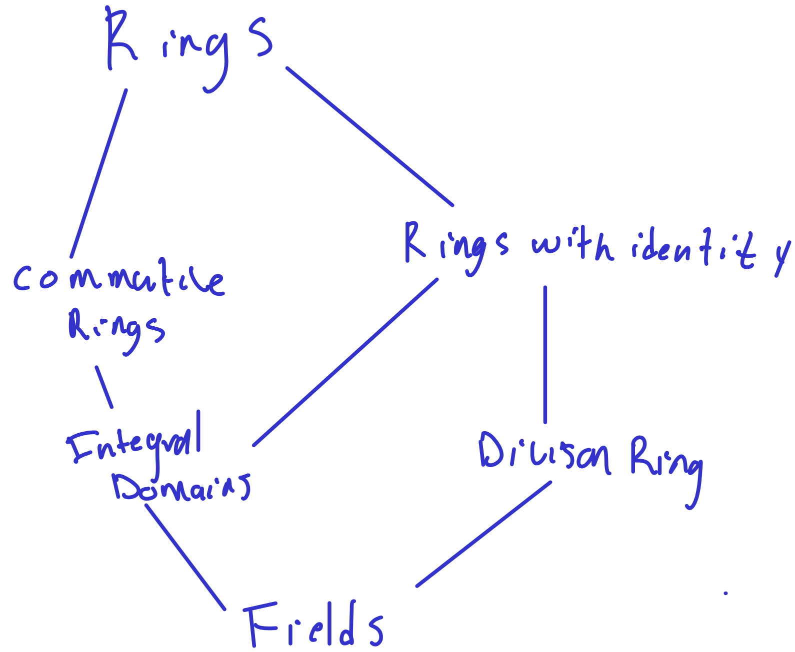
$(R,+)$ Abelian group

- $a+b = b+a$
- $(a+b)+c = a+(b+c)$
- $0 \in R$ s.t. $a+0 = a$
- for all $a \in R$ $\exists -a \in R$ and $a+(-a)=0$
- $(ab)\cdot c = a\cdot(bc)$
- $a(b+c) = ab+ac$
  $(a+b)c = ac+bc$

End of Ring def

Special types of Rings

- Say R has unity or identity if $\exists 1 \in R$ s.t. $1 \neq 0$ and $1a = a\cdot 1 = a$ $\forall a \in R$
- Commutative ring if $ab = ba$ $\forall a,b \in R$
- A commutative ring with identity where if $ab=0$ then either $a=0$ or $b=0$ is called an Integral Domain.
- A ring is a Division Ring if $\forall a \in R$ $a \neq 0$ $\exists$ $a^{-1} \in R$ (unique) s.t. $a^{-1}a = aa^{-1} = 1$. We call a a unit in R
- A commutative Division ring is a feild.



Ex)

$\mathbb{Z}$ - integral Domain Not a feild

\ 1, -1 are only units

Feilds: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$

Ex) $\mathbb{Z}_n$ is a Ring, Not an integral Domain in general

zero divisors.

Ex $\mathbb{Z}_{12}$, $3 \cdot 4 = 12 \bmod 12 = 0$

$\therefore$ Not integral Domain

Def)

$a \in R$ is a zero divisor $(a \neq 0)$ if $\exists\, b \in R$ $(b \neq 0)$

s.t. $a \cdot b = 0$.

Ex) $C[a,b]$ = Continuous real valued functions on $[a,b]$

$f, g \in C[a,b]$ ↑ Commutative ring

$f+g = f(x)+g(x)$, $f \cdot g = f(x)g(x)$

$f = x^2 \quad g = \sin x$

$f+g = x^2 + \sin x$, $f \cdot g = x^2 \sin x$

$\therefore$ a ring $\qquad \therefore$ has unity.

$0 \in C[a,b]$, $1 \in C[a,b]$

Ex) 2×2 Real matrices

$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$AB \neq BA$, matrices A, B.

Ex) The Quaternions are a noncommutative division Ring

$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $j = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $k = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$\mathbb{H} = \{ a\cdot 1 + bi + cj + dk \mid a,b,c,d \in \mathbb{R} \}$ is a ring

$i^2 = j^2 = k^2 = -1$, $ij = k$, $ji = -k$, ...

$\therefore$ not commutative

$$(a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}) + (a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k})$$
$$= (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\mathbf{j} + (d_1 + d_2)\mathbf{k}$$

and

$$(a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k})(a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}) = \alpha + \beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k},$$

where

$$\alpha = a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2$$
$$\beta = a_1b_2 + a_2b_1 + c_1d_2 - d_1c_2$$
$$\gamma = a_1c_2 - b_1d_2 + c_1a_2 - d_1b_2$$
$$\delta = a_1d_2 + b_1c_2 - c_1b_2 - d_1a_2.$$

$(a+ib)^{-1} = \dfrac{a-ib}{a^2+b^2}$

$$(a + bi + cj + dk)(a - bi - cj - dk) = a^2 + b^2 + c^2 + d^2.$$

This element can be zero only if a, b, c, and d are all zero. So if $a + bi + cj + dk \neq 0$,

$$(a + bi + cj + dk)\left(\frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2}\right) = 1.$$

$\therefore (a + bi + cj + dk)^{-1} = \left(\dfrac{a - bi - cj - dk}{a^2+b^2+c^2+d^2} \right)$ $\therefore$ $\mathbb{H}$ is a division Ring But NOT a feild

Proposition Let R be a ring with $a,b \in R$. Then

1) $a \cdot 0 = 0 \cdot a = 0$

2) $a(-b) = (-a)b = -ab$

3) $(-a)(-b) = ab$

Proof: (1)

since 0 is add. identit

by distributive property

$$a0 = a(0+0) = a0 + a0$$

since $-a0 + a0 = 0$ additive inverse.

$$\Rightarrow \quad 0 = a0$$

(2)

distributive property

$$ab + a(-b) = a(b-b) = a \cdot 0 = 0$$

$$a(-b) = -ab.$$

(3)

by (2)

$$(-a)(-b) = -(a(-b)) = -(-ab) = ab$$

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