Polynomial Rings

Let R be a commutive ring with identity 1

Any expression

$$f(x) = \sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + \cdots + a_n x^n$$

$a_i \in R$, $a_n \neq 0$ is a polynomial with coefficents in R and intermedrate x.

a_n - leading coefizant

$a_n x^n$ - leading term

monic if $a_n = 1$

If n is the largest non-negitive s.t

$$a_n \neq 0 \Rightarrow \deg(f) = n$$

$\uparrow$ degree

If no such n exists $\Rightarrow f = 0$

$$\deg(0) = -\infty$$

$$a_0 + a_1 x + \cdots + a_n x^n = b_0 + b_1 x + \cdots + b_m x^m$$

iff $a_i = b_i \ \forall i \geq 0$

$R[x] = \{$ Set of Polynomials in x with coeficents in $R \}$

To show $R[x]$ is a ring define

- Add.

$$(\overset{p(x)}{\overset{\shortparallel}{a_0 + a_1 x + \cdots + a_n x^n}}) + (\overset{q(x)}{b_0 + b_1 x + \cdots + b_m x^m})$$

(say $n \geq m$)

$$= (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

mult.

$$p(x)q(x) = \sum_{i=0}^{m+n} \left(\sum_{k=0}^{i} a_k b_{i-k} \right) x^i$$

(some $a_j, b_j = 0$ perhaps)

Ex] work $\mathbb{Z}_{12}[x]$

$p(x) = 3 + 3x^3$, $q(x) = 4 + 4x^2 + 4x^4$

$p(x) + q(x) = 7 + 4x^2 + 3x^3 + 4x^4$

$p(x)q(x) = 0$

↑ are zero divisors

Theorem] Let R be a comm. ring with identity.

Then $R[x]$ is a comm. ring with identity.

Proof:

Show $R[x]$ is an abelian group with add.

- $f(x) = 0$
- Add inverse of $p(x) = \sum a_i x^i$ is $-p(x) = \sum (-a_i) x^i$

Show mult is associtive

$$p(x) = \sum_{i=0}^{m} a_i x^i \quad q(x) = \sum_{i=0}^{n} b_i x^i, \quad r(x) = \sum_{i=0}^{v} c_i x^i$$

$$\left(p(x) \cdot q(x)\right) \cdot r(x) = \left[\left(\sum_{i=0}^{m} a_i x^i\right)\left(\sum_{i=0}^{n} b_i x^i\right)\right]\left(\sum_{i=0}^{v} c_i x^i\right)$$

$$= \left[\sum_{i=0}^{m+n}\left(\sum_{j=0}^{i} a_j b_{i-j}\right) x^i\right] \cdot \left(\sum_{i=0}^{v} c_i x^i\right)$$

$$= \sum_{i=0}^{m+n+v}\left(\sum_{j=0}^{i}\left[\sum_{k=0}^{j} a_k b_{j-k}\right] c_{i-j}\right) x^i$$

$$= \sum_{i=0}^{m+n+v}\left(\sum_{j+k+l=i} a_j b_k c_l\right) x^i$$

$$= \sum_{i=0}^{m+n+v}\left[\sum_{j=0}^{i} a_j \sum_{k=0}^{i-j} b_k c_{i-j-k}\right] x^i$$

$$= \overbrace{\left(\sum_{i=0}^{m} a_i x^i\right)}^{=p(x)}\left[\sum_{i=0}^{n+v}\left(\sum_{j=0}^{i} b_j c_{i-j}\right) x^i\right]$$

$$= p(x) \cdot \left[\left(\sum b_i x^i\right) \cdot \left(\sum c_i x^i\right)\right]$$

$$= p(x)\left[q(x) \cdot r(x)\right].$$

∎

Prop] Let $p(x), q(x) \in R[x]$ where R is an integral domain. Then $\deg(p(x) \cdot q(x)) = \deg(p(x)) + \deg(q(x))$.

and $R[x]$ is an integral Domain.

Proof: Let $p, q \in R[x]$, $p \neq 0$, $q \neq 0$

$$p = a_m x^m + \cdots + a_1 x + a_0$$

$$q = b_n x^n + \cdots + b_1 x + b_0$$

$\deg(p) = m$, $\deg(q) = n$

Lead term of $p(x)q(x) = a_m b_n x^{m+n}$ since $a_n \neq 0$ and $b_n \neq 0$ and R is an integral Domain $\Rightarrow$ $a_m b_n \neq 0$

$\therefore \deg(p \cdot q) = m+n$

$p(x)\,q(x) \neq 0$ (whenever p and q are non zero)

$\therefore$ $R[x]$ is an integral Domain ∎

Multivariant polynomial Rings

i.e. $x^2 - 3xy + 2y^3$

· $R[x]$ is a commutative ring with 1

$\therefore$ $(R[x])[y]$ is a comm. ring with 1, so is

typical = $b_m(x)y^m + \cdots + b_1(x)y + b_0(x)$

$(R[y])[x]$ = typical element $a_n(y)x^n + a_{n-1}(y)x^{n-1} + \cdots + a_1(y)x + a_0(y)$

Show $(R[x])[y] \cong (R[y])[x]$

$R[x, y]$.

Corr) $R[x_1, \ldots, x_n]$ is a commutative ring with 1.

Thm Let R be a commutative ring with 1. Let $\alpha \in R$

Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$

$\phi_\alpha : R[x] \longrightarrow R$ defined by

$\phi_\alpha(p(x)) = p(\alpha) = a_n \alpha^n + \cdots + a_1 \alpha + a_0$

is a ring hom. We call ϕ_α the evaluation hom.

Proof: $p(x) = \sum a_i x^i$, $q(x) = \sum b_i x^i$

$\phi_\alpha(p(x) + q(x)) = (p+q)(\alpha) = \sum (a_i + b_i)\alpha^i = p(\alpha) + q(\alpha) = \phi_\alpha(p) + \phi_\alpha(q)$

$= \sum a_i \alpha^i + \sum b_i \alpha^i$ //

$$\begin{aligned}
\phi_\alpha(p(x))\,\phi_\alpha(q(x)) &= p(\alpha)\,q(\alpha) \\
&= \left(\sum_{i=0}^{n} a_i \alpha^i\right)\left(\sum_{i=0}^{m} b_i \alpha^i\right) \\
&= \sum_{i=0}^{m+n}\left(\sum a_k b_{i-k}\right)\alpha^i \\
&= p\cdot q(\alpha) \\
&= \phi_\alpha(p(x)q(x))
\end{aligned}$$

$\blacksquare$