Some important things to understand (note this list is not exhaustive):

-What is a subgroup?

-How do you show something is a subgroup?

-What is a coset?

-What is a normal subgroup?

-How do you show something is a normal subgroup?

-What is an abelian group?

-What is a cyclic group?

-Lagrange's Theorem

-The index of a subgroup (see Lagrange's theorem, cosets chapter, etc.)

-Euler's Theorem (and Euler's phi function) and Fermat's Little Theorem

-What is an isomorphism?

-What is a homomorphism?

-The 3 Isomorphism theorems and how to use them.
The first one can be especially useful (note you may find the form in Mondays lectures notes easier to apply than the way it is stated in the book, it is same, I just summarized what the book has)

-How to show a map is a homomorphism

-How to show a map is an isomorphism (both using an isomorphism theorem and directly from the definition)

- What is the kernel of a homomorphism?

-How are Normal subgroups and kernels of homomorphisms related? i.e why are they the same...

-General stuff about how to write proofs rigorously etc.

Find all Subgroups

$ij = k, \quad ik$

$$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad k = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$jk = i$

$ik = \pm j$

$$Q_8 = \{ i, j, k, -i, -j, -k, 1, -1 \}$$

$ij \stackrel{?}{=} ji$

$$H = \{1, -1, i, -i\}$$

4 subgroups

1 order 2

3 order 4

$$H_2 = \{1, -1\}$$

Normal Subgroup N of G

$$gNg^{-1} \subset N \quad \forall g \in G$$

$$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$ij = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = k$$

$$ji = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -k$$

$$Q_8/H_2 = \{H_2, iH_2, jH_2, kH_2\}$$

$$Q_8 = \{i, j, k, -i, -j, -k, 1, -1\}$$

$$H_2 = \{1, -1\}$$

$$Q_8/H_1 = \{H_1, jH_1\}$$

$$jH_1 = kH_1 = \{k, -k, j, -j\}$$

$$= \{j, -j, k, -k\}$$

~~$H_1 = \{1, -1, i, -i\}$~~

If $|G| = 2n$ then $|H| = n$ a subgroup is normal

$G = G_1 \cup G_2$

subgroup of order n of $|G| = 2n$

2nd iso theorem)

Let H be a subgroup of G. Let N be a normal subgroup of G. Then HN is a subgroup of G, $H \cap N$ is a normal subgroup and

$$H/(H \cap N) \cong HN/N.$$

Proof:

Show $HN = \{hn : h \in H, n \in N\}$ is a subgroup of G.

- $e \in HN$ since $e \in H$, $e \in N$
- Show closure $\qquad \forall h \in H$

Let $h_1 n_1, h_2 n_2 \in HN$ since N is normal $\Leftrightarrow h^{-1} N h \subset N$

$\Rightarrow h^{-1} n h \in N$

$\forall n \in N$

$$h_1 n_1 h_2 n_2 = \underbrace{h_1 h_2}_{\in H} \underbrace{\overbrace{h_2^{-1} n_1 h_2}^{\in N}}_{\in N} \underbrace{n_2}_{\in N}$$

gNg^{-1} – congruacy class of N w.r.t g

$\therefore$ closed

- Inverses

$$(hn)^{-1} = n^{-1} h^{-1} = \underbrace{h^{-1}}_{\in H} h \underbrace{n^{-1} h^{-1}}_{\in N} \quad \therefore (hn)^{-1} \in HN.$$

Now prove $H \cap N$ is normal in H.

$h \in H, \quad n \in H \cap N$

$\Rightarrow \quad h^{-1} n h \in H \quad h^{-1} n h \in N$ since N is normal

$$h^{-1} n h \in H \cap N$$

$$\Updownarrow$$

$$h^{-1} H \cap N h \subseteq H \cap N$$

Rest of steps

- Define $\phi : H \longrightarrow HN/N$

$h \longmapsto hN$ anything in HN/N

$n \in N \therefore nN = N$

- ϕ is onto since $hnN = hN = \phi(h)$. $\Rightarrow HN/N = \phi(H)$

$\therefore$ By 1st isomorphism theorem

$$\phi(H) \cong H / \ker \phi$$

$\ker \phi = \{ h \in H \mid \phi(h) = N \} = H \cap N$

if $\phi(h) = hN = N \Leftrightarrow h \in N$.

$$\therefore HN/N \cong H / H \cap N.$$