Third Iso. Therem

Let R be a ring and I and J be ideals of R where $J \subset I$. Then

$$R/I \cong \frac{R/J}{I/J}$$

Correspondence Theorem

S a subring of R

Let I be an ideal in a ring R. Then $S \to S/I$ is a 1-1 correspodncc

$$\{\text{Subrings } S \text{ of } R \text{ s.t. } I \subseteq S\} \xleftrightarrow[S \to S/I]{} \{\text{Subrings of } R/I\}$$

$$\{\text{ideals of } R \text{ cont. } I\} \longleftrightarrow \{\text{ideals of } R/I\}$$

Maximal and Prime Ideals

decide when R/I is
a feild?
an integral domain?

Def | A proper ideal M of a ring R is called a maximal ideal if:

- M is not a proper subset of any ideal of R other than R.

$\Updownarrow$ equivilently

- M is maximal if for any ideal I of R s.t $M \subsetneq I \Rightarrow I = R$.

(Think: Feilds R only have ideas $\{0\}$ and R)

Theorem | Let R be a commutitive ring with $1 \in R$ and let M be an ideal in R. M is maximal if and only if R/M is a feild.

Proof:

Let M be a maximal ideal in R.

R commutitive $\Rightarrow$ R/M commutive

$1 + M$ is the identity in R/M

Show inverses exist for non-zero ele. in R/M

If $a + M \neq 0 + M$ in R/M

$\Rightarrow$ $a \notin M$. Fix (arbitary) $a + M \neq 0 + M \in R/M$

Let $I = \{ ra + m \mid r \in R, m \in M \}$

Show I is an ideal:

- I non-empty since $0 \cdot a + 0 = 0 \in I$
- Let $r_1 a + m_1 \in I$ $r_2 a + m_2 \in I$

$$r_1 a + m_1 - (r_2 a + m_2) = \underbrace{\overset{\in R}{(r_1 - r_2)} a + \overset{\in M}{(m_1 - m_2)}}_{\in I}$$

- For any $\tilde{r} \in R$

$$\tilde{r}(ra + m) = \underbrace{\overset{\in R}{\tilde{r}r} a + \overset{\in M}{\tilde{r} m}}_{\in I}$$

$\therefore$ I is an ideal

By construction $M \subsetneq I$ Since $a \in I$ $a \notin M$

M is maximal $\Rightarrow$ $I = R$ Now $1 \in R$ and $I = R$

$\therefore$ $\exists\, b \in R$ s.t $\Rightarrow$ $1 \in I$

$\exists\, m \in M$

$ba + m = 1$ (remember we started with $a \notin M$ giving $a + M$ in R/M)

$\therefore$ $1 + M = (ba + m) + M = ab + M = ba + M = (a + M)(b + M)$

$= (b + M)(a + M)$

$\therefore$ $(b + M) = (a + M)^{-1}$ in R/M

$\therefore$ R/M is a feild.

Now suppose M is an ideal, R/M is a feild.

$\Rightarrow$ $0 + M$, $1 + M \in R/M$

$\therefore$ $M \neq R$ i.e. M is a proper ideal of R, $M \subsetneq R$

Let I be any ideal of R s.t. $M \subsetneq I$
Show $I = R$.

Pick some $a \in I$, $a \notin M$

$\Rightarrow$ $a + M \neq 0 + M$

$\Rightarrow$ $\exists$ some $b + M$ s.t. $(a+M)(b+M) = (b+M)(a+M)$

$$(ab+M) = 1 + M$$

$\Rightarrow$ $\exists$ $m \in M$ s.t. $ab + m = 1$

but $\underbrace{ab}_{\in I \text{ as } a \in I} + \underbrace{m}_{\in I} \in I$

$\Rightarrow$ $1 \in I$ $\Rightarrow$ $r \cdot 1 = r \in I$ $\forall r \in R$

$\Rightarrow$ $I = R$. $\blacksquare$

Ex) $p\mathbb{Z}$ is maximal in $\mathbb{Z}$ for p prime

Since $\mathbb{Z}/p\mathbb{Z}$ is a feild.

Def) A proper ideal P in a commutive ring R is called a prime ideal if whenever $ab \in P$ then either $a \in P$ or $b \in P$.

Ex) $P = \{0, 2, 4, 6, 8, 10\}$ is a prime ideal of $\mathbb{Z}_{12} \cong \mathbb{Z}/12\mathbb{Z}$. $P = 2\mathbb{Z}$ in $\mathbb{Z}_{12}$

Proposition) Let R be a commutive ring. $1 \in R$, $1 \neq 0$

Then P is a Prime ideal in R if and only if R/P is an integral domain.

Proof: First let P be an ideal of R

R/P is an int. dom.

Suppose $ab \in P$ then

$$(a+P)(b+P) = ab + P = 0 + P \quad \text{in } R/P$$

But R/P is an integral domain

either $a+P = 0+P$ or $b+P = 0+P$

$\Rightarrow$ either $a \in P$ or $b \in P$

$\Rightarrow$ P is a prime ideal.

Now suppose P is prime, Show R/P has no zero divisors.

$$(a+P)(b+P) = 0 + P = P$$

$$||$$

$$ab + P = 0 + P$$

$$ab \in P$$

$\Rightarrow$ either $a \in P$ or $b \in P$

$\Rightarrow$ either $a+P = 0+P$ or $b+P = 0+P$

$\Rightarrow$ R/P is an integral domain.

$\blacksquare$

Every feild is in particular an integral domain

corr) Every maximal ideal in a commutative ring with identity is also a prime ideal.