Proposition

The Kernel of any ring homomorphism $\phi: R \to S$ is an ideal in R (i.e. $\ker(\phi)$ is an ideal in R).

Proof: From groups we know tha $\ker\phi$ is additive subgroup of R

Let $r \in R$, $a \in \ker(\phi)$ show $ar \in \ker(\phi)$ and $ra \in \ker(\phi)$

$$\phi(ar) = \underbrace{\phi(a)}_{=0 \text{ since } a \in \ker(\phi)} \cdot \phi(r) = 0 \cdot \phi(r) = 0 \quad \therefore ar \in \ker\phi$$

$$\phi(ra) = \phi(r)\underbrace{\phi(a)}_{=0} = \phi(r) \cdot 0 = 0 \quad \therefore ra \in \ker\phi.$$

∎

R/I = Quotient Ring for R a ring, I an ideal

Theorem: Let I be an ideal of R. The factor R/I is a ring with multiplication given by

$$(r+I)(s+I) = rs + I.$$

[we already know R/I is an abelian group under addition $(r+I)+(s+I) = (r+s)+I$ sines an ideal I is also a normal subgroup.]

Proof:

Let $s+I$, $r+I \in R/I$ show that the mult. is well defined

Let $r' \in r+I$ $s' \in s+I$ Show $r's' \in rs+I$

$r' \in r+I \Rightarrow \exists a \in I$ s.t $r' = r+a$

$s' \in s+I \Rightarrow \exists b \in I$ s.t $s' = s+b$

$$r's' = (r+a)(s+a) = rs + \underbrace{as + rb + ab}_{\in I} \quad \text{Since } a,b \in I$$

$$r's' \in rs + I$$

<u>Distributivity</u> say $r+I,\ s+I,\ w+I \in R/I$

Show

$$(r+I)\big((s+I)+(w+I)\big) = (r+I)\big((s+w)+I\big)$$
$$= r(s+w) + I$$
$$= rs + rw + I$$
$$= (rs+I) + (rw+I)$$
$$= (r+I)(s+I) + (r+I)(w+I).$$

Associativity similar

$\blacksquare$

<u>Theorem</u>

Let I be an ideal of R. The map $\psi : R \to R/I$ defined by $\psi(r) = r+I$ is a ring homomorphism of R onto R/I and $\ker(\phi) = I$.

<u>Proof:</u> From Groups we know

$\psi : R \to R/I$ is a surjective group hom.

Show ψ is a ring hom. Let $r, s \in R$

$$\psi(r)\psi(s) = (r+I)(s+I) = rs+I = \psi(rs)$$

$\blacksquare$

$\psi : R \to R/I$ is called the natural/canonical Ring hom.

ker ϕ is an ideal

Theorem (First iso. Theorem for rings)

Let $\phi : R \to S$ be a ring homomorphism. Let $\psi : R \to R/\ker(\phi)$ be the canonical hom. Then there exists a unique isomorphism

$$\eta : R/\ker\phi \longrightarrow \phi(R) \quad \text{s.t.} \quad \phi = \eta \circ \psi$$

In particular $\boxed{\phi(R) \cong R/\ker(\phi)}$

$\phi(R)$ is a subring of S.

$$\begin{tikzcd} R \arrow[rr, "\phi"] \arrow[dr, "\psi"'] & & S \\ & R/\ker(\phi) \arrow[ur, "\exists!\,\eta"'] & \end{tikzcd}$$

s.t. Diagram commutes.

Proof: Let $K = \ker\phi$. By the 1^{st} iso. theorem for Groups $\exists$ a unique (well defined) group hom.

$$\eta : R/K \longrightarrow \phi(R)$$
$$r+K \longmapsto \phi(r)$$

(for add. groups R/K, R, S)

we need to show this extends to a ring hom.

$$\eta\big((r+K)(s+K)\big) = \eta(rs+K)$$
$$= \phi(rs)$$
$$= \phi(r)\phi(s)$$
$$= \eta(r+K)\,\eta(s+K)$$

$\therefore$ η is a ring hom. and is unique $\therefore$ we are done $\blacksquare$

Theorem (Second Iso. Theorem)

Let I be a subring of a ring R and J to be an ideal of R. Then $I \cap J$ is an ideal of I and

internal direct product i.e

$$I/(I \cap J) \cong (I+J)/J .$$

$a+b$ st $a \in I$, $b \in J$

Proof:

- $I+J$ is a subring of R.

  we know $I+J$ is an abelian subgroup.

  Let $a, a' \in I$, $b, b' \in J$

  $$(a+b)(a'+b') = \underbrace{\overset{\in I}{aa'} + \overbrace{ba' + ab' + bb'}^{\in J \text{ since } J \text{ is an ideal}}}_{\in I+J}$$

- Show J is an ideal of $I+J$ $\quad c \in J$

  Let $a \in I$, $b \in J$ for any $a+b \in I+J$ show $(a+b)c \in J$, $c(a+b) \in J$

  $$(a+b)c = \overset{\in J}{ac} + \overset{\in J}{bc}$$

  $$c(a+b) = \overset{\in J}{ca} + \overset{\in J}{cb}$$

  $\therefore$ J is an ideal of $I+J$.

Now define $\phi : I \longrightarrow (I+J)/J$

$$a \longmapsto a+J \quad (a \in I)$$

Show ϕ is a hom. of rings. Let $a_1, a_2 \in I$

$$\phi(a_1+a_2) = a_1+a_2+J = (a_1+J)+(a_2+J)$$
$$= \phi(a_1)+\phi(a_2)$$

$$\phi(a_1a_2) = (a_1a_2+J) = (a_1+J)(a_2+J)$$
$$= \phi(a_1)\,\phi(a_2)$$

(well defined follows since $\psi: R \longrightarrow R/J$ is well defined)

ϕ is onto since $\forall\, a \in I,\ b \in J$

$$\underbrace{a+b+J}_{\in (I+J)/J}$$
$$= a+J = \phi(a)$$

$$\ker(\phi) = \{a \in I \mid \phi(a) = 0+J\}$$
$$= \{a \in I \mid a \in J\}$$
$$= I \cap J \quad \therefore\ I\cap J \text{ is an ideal}$$

$\therefore\ \phi: I \longrightarrow (I+J)/J$ is an onto ring hom.

$$\phi(I) \cong I/\ker(\phi) = I/(I\cap J)$$
$$\|$$
$$(I+J)/J \cong$$

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